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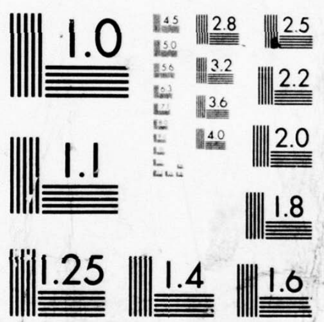

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STUDIES AND RESEARCH ON APPLIED MECHANICS
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THE SUPERSONIC FLOW AROUND SYMMETRICALLY TORSIONED AND CURVED DELTA WINGS, TAKING INTO CONSIDERATION THE SEPARATION OF FLOW AT THE LEADING EDGES

by ELIE CARAFOLI and STEFAN STAICU
Institute of FLUID MECHANICS
Academy of the Socialist Republic of Romania

In this work the supersonic flow around deformed thin delta wings is studied, having the distribution of incidences symmetrical and varying proportionally with x_1 and x_2. Taking into consideration the separation of flow at the subsonic leading edges, the distribution of pressure and and aerodynamic characteristics of the wing are determined, through the intermediary of an imaginary wing, equivalent from an aerodynamic point of view, with the real wing.

1. PRELIMINARY CONSIDERATIONS

In the present work we will study the supersonic flow with the separation of flow at the leading edges of deformed thin delta wings, in such a way that the incidence is symmetrical in reference with the axis of symmetry and varies proportionally with x_1 and x_2 (fig. 1).

As with the plane thin delta wing, with constant incidence (1), (9), in this case the flow separates at the leading edges, creating as well a vortex layer which winds itself approximately in the form of two horns, which are then transformed for the most part of the vortex generating intensities into two concentrated vortex nuclei, situated symmetrically in reference with axis Ox_1 (fig. 1) and defined by the coordinates c and t. The system of two concentrated vortexes considered will bring changes on the flow.

The resulting motion, which becomes more complicated now, will be studied on the basis of the theory of motion of the second order. Towards this goal, we will follow the road used in previous papers (1), (3) and (9), where solutions were given for thin delta wings with constant incidence and with antisymmetrical constant incidence respectively (forced antisymmetry) in reference with Ox_1, which led us to conical motion as a matter of fact.

In this manner, we will allow that the effect of the falling off of flow at the leading edges of the wing and therefore of the form of the vortex nuclei on the higher side of the wing is that of changing the vertical velocities and axises of disturbance, becoming finite at the edges. We can equate this complex effect with that of a wing with a variation corresponding to the incidence

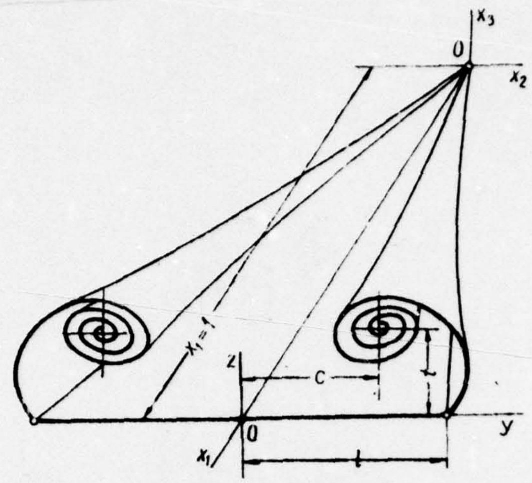

Fig. 1

or with a variation of the vertical velocities of disturbance. Through this we substituted the horizontal velocities of disturbance ~~among~~ for an equivalent distribution of vertical velocities.

We will obtain a symmetrical distribution of vertical velocities on the higher side of the wing, which will correspond in this way to an imaginary thin delta wing, with variable incidence, having at the same time a finite velocity at the leading edges. It will be allowed that the real thin wing, which has in a certain way finite velocities at the edges through the effect of the separation of flow, is equivalent, from an aerodynamic point of view, with an imaginary wing, having the same variation of incidences which we defined above.

Since the incidence is variable on the surface of the wing, the axis on which the horn winds will be a curve, and the vortex

generating intensity of the nucleus is variable along the axis, proportional with the square of the span of the wing. For simplification, in the following, the axis on which the vortex nucleus is situated is considered a straight line.

Noting further

$$w'_u = -\alpha'_u U_\infty, \quad w'_l = -\alpha'_l U_\infty \quad (w_0 = -\alpha_0 U_\infty, \ w_1 = -\alpha_1 U_\infty) \quad (1$$

the vertical velocities and the incidences w'_u, α'_u on the higher side, respectively, w'_l, α'_l on the lower side, for the imaginary thin wing we can write the relation

$$\int_0^l w'_u \, dy + \int_0^l w'_l \, dy = 2wl \quad (w = -\alpha U_\infty), \quad (2$$

where the velocity w on the real wing, in conical motion of the second order, is the homogenous function of the first order,

$$w = w_{10}x_1 \pm w_{01}x_2 = x_1(w_{10} \pm w_{01}y) = -x_1(\alpha_{10} \mp \alpha_{01}y)U_\infty \left(y = \frac{x_2}{x_1}\right), \quad (3)$$

in which the term $w_{10}x_1$ corresponds to the incidence with natural symmetry (the curved delta wing), and $\pm w_{01}x_2$ corresponds to the incidence with forced symmetry (the torsioned delta wing), the sign (+) being considered for the right part and (-) for the left.

In the same way as in (1), (3) and (9), we will split the imaginary wing into three wing components corresponding to the distribution of vertical velocities above.

1) The thin wing with the variation of chosen corresponding

incidence, in such a way as to respect somewhat the significance of the phenomena of pressure changes at the leading edges. In this way a thin imaginary wing with finite velocities at the leading edges, and equal and opposed direction on the two sides, higher and lower, is obtained.

2) The wing of symmetrical thickness, having the slope variable in the same way as the incidence of the first wing. This wing combined with that from 1) has different pressures on the two sides, as happens in reality.

3) The third wing will have a symmetrical thickness, with the slope also symmetrical, however in such a way that, combined with the wing from 2), an average nought slope will be obtained, corresponding to a real thin wing.

2. DETERMINATION OF THE AXIS OF DISTURBANCE VELOCITIES

In continuation we will determine, for the three imaginary wing components, the axis of disturbance velocities, being necessary for the calculation of the distribution of pressures and aerodynamic characteristics of the resulting imaginary wings, which are presupposed to be the same as those of the real thin delta wings, having the incidence defined by (3).

However, the motion around the wing being conical of the second order, we will utilize the same methods used, considering in this sense the section obtained in normal plane Oyz for the direction of undisturbed flow U_∞ and having the coordinates

$$y = \frac{x_2}{x_1}, \quad z = \frac{x_3}{x_1}, \tag{4}$$

axes Oy and Oz being parallel with Ox_2 and Ox_3 respectively. We will further make a similar transformation with that given by Busemann (fig. 2):

$$\mathfrak{y} = \frac{y}{1 - B^2 z^2}, \quad \mathfrak{z} = \frac{z\sqrt{1 - B^2(y^2 + z^2)}}{1 - B^2 z^2} \quad (x = \mathfrak{y} + i\mathfrak{z}), \tag{5}$$

obtaining a plane which has the property of keeping the track of the wing ($\mathfrak{Y}=y$, $z = \mathfrak{z} = 0$) in the true magnitude.

As we know, in this plane the first derivatives of the velocities of disturbance u, v, w are harmonic functions and the conjugated functions can be associated respectively in such a way as to obtain analytic functions of complex variables

$$x = \mathfrak{y} + i\mathfrak{z}. \tag{6}$$

We will study further each wing defined above in turn.

1). The thin wing. As a result of the effect of the two vortex nuclei, the vertical velocity on the real wing is modified, as well as on the first wing component defined above. Thus, for the points contained between ($-s<y<s$) from the track of the wing from the plane $x = \mathfrak{Y} + i\mathfrak{z}$ (6), the vertical velocity is the same as the form given by (3):

$$\frac{w'}{x_1} = w_{10}^{(0)} \pm w_{01}^{(0)} y = -(\alpha_{10}^{(0)} \mp \alpha_{01}^{(0)} y) U_\infty, \tag{7}$$

where parameters $w_{10}^{(0)}$ and $w_{01}^{(0)}$ correspond to the abscissa $y = s$, and for the intervals $(-l,-s)$ and (s,l) we will write

$$\frac{w'}{x_1} = w'_{10}(y) \pm w'_{01}(y) \; y = -\left(\alpha'_{10}(y) \mp \alpha'_{01}(y) \, y\right) U_\infty, \tag{8}$$

so that at the edges of the wing we will obtain

$$\frac{w^{(1)}}{x_1} = w_{10}^{(1)} \pm w_{10}^{(1)} l. \tag{9}$$

The continuing variation of the vertical velocities (or, more precisely, of the parameters $w_{10}(y)$ and $w_{01}(y)$) or the continuing incidences corresponds to the distribution of elementary edges situated on the wing in the interior of the interval considered, which give in each point $y = \eta$ the elementary drop

$$\left(\frac{dw'_{10}}{dy} \pm \frac{dw'_{01}}{dy} y\right)_{y=\eta} d\eta. \tag{10}$$

Keeping in mind previous papers (2), (4), the contribution in the expression of the axes of disturbance velocities, of elementary edges situated in points $y = \eta$ on the thin wing with subsonic edges ($l < 1/B$), through the application of similar hydrodynamic methods, will be

$$d\mathcal{U}_1^{(e)} = (q_{20}^{(e)} + x\, q_{21}^{(e)}) \cosh^{-1} \sqrt{\frac{(l+\eta)(l-x)}{2l(\eta - x)}}\, d\eta. \tag{11}$$

In a similar way, for the contribution of elementary edges situated on the left of the origin, $y = -\eta$, we will have

$$d\mathcal{U}_1^{(e)} = (q_{20}^{(e)} + x\, q_{21}^{(e)}) \cosh^{-1} \sqrt{\frac{(l+\eta)(l+x)}{2l(\eta + x)}}\, d\eta. \tag{12}$$

These contributions of the edges in the expressions of axes of disturbance and vertical velocities on the wing are achieved placing some singularities of the second order (in this case - sources) on the track of the wing in plane $x=Y+iz$ (6).

In this way we succeed in acquiring in the calculations the effect of the vortex sheets which fall off from the edges and of the considered concentrated vortex nuclei, through out the continuing distribution of sources.

Next we will allow a simpler form of division of sources, which will satisfy conditions imposed by the problem, for obtaining concommitant axes of disturbance and vertical velocities indicated above on the basis of observations and conforming with experimental results.

Thus, we will chose a liniar variation of intensities of the sources

$$q' = (q^{*(n)} + q^{*(j)})\left(1 - \frac{\eta}{l}\right), \tag{13}$$

where the constants $q^{*(n)}$ and $q^{*(j)}$ correspond to natural symmetries forced respectively, of the incidences. Since the flow is conical of the second order, as seen in (11), (12) we will write

$$q'_{20} = q^*_{20}\left(1 - \frac{\eta}{l}\right), \quad q'_{21} = q^*_{21}\left(1 - \frac{\eta}{l}\right) \quad (s \leqslant \eta \leqslant l). \tag{14}$$

keeping in mind these divisions of sources, starting from (11) and (12), we will consider the contributions of all the elementary distributed edges, as well as on the subsonic edges (2), in order to obtain the following expression of the axes of disturbance vleocities of the first wing components:

$$u_{11} = \frac{1}{x_1}\,\mathcal{U}_1 = \frac{A_{20} + A_{22}\,x^2}{\sqrt{l^2 - x^2}} + \frac{2}{\pi}\int_0^l (q'_{20} + q'_{21}\,x)\cosh^{-1}\sqrt{\frac{(l+\eta)(l-x)}{2l(\eta - x)}}\,d\eta +$$
$$+ \frac{2}{\pi}\int_0^l (q'_{20} - q'_{21}\,x)\cosh^{-1}\sqrt{\frac{(l+\eta)(l+x)}{2l(\eta + x)}}\,d\eta. \qquad (15)$$

Introducing q'_{20} and q'_{21} in (14) and making the calculations we obtain

$$\frac{1}{x_1}\,\mathcal{U}_1 = \frac{A_{20} + A_{22}\,x^2}{\sqrt{l^2 - x^2}} - \frac{1}{\pi}\Big\{(q^*_{20} + q^*_{21}\,x)\,(s - x)\left(1 - \frac{s + x}{2l}\right)\cosh^{-1}\frac{l^2 - sx}{l(s - x)} +$$
$$+ (q^*_{20} - q^*_{21}\,x)\,(s + x)\left(1 - \frac{s + x}{2l}\right)\cosh^{-1}\frac{l^2 + sx}{l(s + x)} +$$
$$+ \left[q^*_{20}\,l\left(\sqrt{1 - \frac{s^2}{l^2}} - 2\cos^{-1}\frac{s}{l}\right) + q^*_{21}\,x^2\cos^{-1}\frac{s}{l}\right]\sqrt{1 - \frac{x^2}{l^2}}\Big\}, \qquad (16)$$

in which A_{20}, A_{22}, q^*_{20}, q^*_{21} are some constants which will be determined below.

2). The wing of symmetrical thickness, with the slope equal with the incidence of the first thin wing. Through the introduction of this wing of symmetrical thickness the accentuated peaks of pressure on the lower side of the wing are removed, where the distribution of pressure, obtained through the superpositioning with the thin wing 1)., is different from that on the higher side. Following the general method of conical motion (2), (4), corresponding to a wing of symmetrical thickness, with the slope

defined by the same distributions of sources given by the relations (13) and (14), we will write for the axis of disturbance velocity the following expression:

$$\mathcal{U}_{1t} = \frac{1}{x_1}\mathcal{U}_t = \frac{2}{\pi}\int_s^l (q'_{20} + q'_{21}x)\cosh^{-1}\sqrt{\frac{(1+B\eta)(1-Bx)}{2B(\eta - x)}}\,d\eta + $$
$$+\frac{2}{\pi}\int_s^l (q'_{20} - q'_{21}x)\cosh^{-1}\sqrt{\frac{(1+B\eta)(1+Bx)}{2B(\eta + x)}}\,d\eta + L, \qquad (17)$$

which, in the course of the accomplishment of the calculations becomes

$$\frac{1}{x_1}\mathcal{U}_t = \frac{1}{\pi}\Bigg\{(q^*_{20} + q^*_{21}x)\left[(l-x)\left(1-\frac{l+x}{2l}\right)\cosh^{-1}\frac{1-B^2lx}{B(l-x)} - \right.$$
$$\left. -(s-x)\left(1-\frac{s+x}{2l}\right)\cosh^{-1}\frac{1-B^2sx}{B(s-x)}\right] +$$
$$+(q^*_{20} - q^*_{21}x)\left[(l+x)\left(1-\frac{l-x}{2l}\right)\cosh^{-1}\frac{1+B^2lx}{B(l+x)} - \right.$$
$$\left. -(s+x)\left(1-\frac{s-x}{2l}\right)\cosh^{-1}\frac{1+B^2sx}{B(s+x)}\right] +$$
$$+\frac{1}{Bl}\left[2l\left(\sin^{-1}Bl - \sin^{-1}Bs + \frac{\sqrt{1-B^2l^2} - \sqrt{1-B^2s^2}}{2Bl}\right)q^*_{20} - \right.$$
$$\left. -(\sin^{-1}Bl - \sin^{-1}Bs)\,x^2\,q^*_{21}\right]\sqrt{1-B^2x^2}\Bigg\} + L, \qquad (18)$$

where the term L is due to the slope of the leading edge:

$$\frac{\gamma^{(1)}}{x_1} = \gamma^{(1)}_{10} + \gamma^{(1)}_{01}\,l. \qquad (19)$$

3). The wing of symmetrical thickness compensating for slope. Through the effect of the wing from point 2)., the resulting wing became the wing of "symmetrical thickness". In order to compensate this work and to put us in accord with reality, we will introduce a new distribution of source of a certain form, which

will bring the wing back to a mean nought thickness. The variation of vertical velocities w_{10}^{*}, equal with $-w_{10}^{(1)}$, respectively $\Upsilon_{10}^{(1)} = -\alpha_{10}^{(1)}$ at the extremity of the wing (where the indices (1) correspond to the two marginal parts of the wing) will correspond with "the wing compensating for slope" of symmetrical thickness which will cancel the slope and the effect L of the slope $\frac{\Upsilon^{(1)}}{X_1}$ from the extremity of the wing 2). In this way the term which produces a velocity tending towards infinity at the leading edge is eliminated according to a logrithmic expression. The distribution of sources $q^{''}$ will be necessary to create, at the same time, on the lower side of the wing, a distribution of pressure without accentuated peaks, approximately constant with the exception of the regions near the leading edges.

Then, for simplification, we will chose the following functions for the distribution of sources:

$$q_{20}^{''} = k_{20}\,\eta(1 + k_{10}\,\eta), \tag{20a}$$

$$q_{21}^{''} = k_{21}\,\eta(1 + k_{11}\,\eta) \quad (0 \leqslant \eta \leqslant l). \tag{20b}$$

Thus, we got those two "thick wing" components, 2). and 3)., in order to be reduced to one with the slope variable, having therefore the mean nought slope.

For the "compensating wing", the function of the axes velocities U_{1c} will be, similar with (17), the following:

$$\mathcal{U}_{1c} = \frac{1}{x_1}\,\mathcal{U}_c = \frac{2}{\pi}\int_0^l (q_{20}^{''} + q_{21}^{''}\,x)\cosh^{-1}\sqrt{\frac{(1+B\eta)(1-Bx)}{2B(\eta - x)}}\,d\eta +$$

$$+\frac{2}{\pi}\int_0^l (\ddot{q}_{20}-\ddot{q}_{21}\,x)\cosh^{-1}\sqrt{\frac{(1+B\eta)(1+Bx)}{2B(\eta+x)}}\,\mathrm{d}\eta - L. \tag{21}$$

In the course of the calculations we are driven towards the expression

$$\begin{aligned}
\frac{1}{x_1}\mathcal{U}_e = {} & \frac{k_{20}}{2\pi B^2}\Bigg[B^2(l^2-x^2)\left(\cosh^{-1}\frac{1-B^2lx}{B(l-x)}+\cosh^{-1}\frac{1+B^2lx}{B(l+x)}\right)+ \\
& + 2B^2x^2\cosh^{-1}\frac{1}{Bx}+2\left(1-\sqrt{1-B^2l^2}\right)\sqrt{1-B^2x^2}\Bigg]+ \\
& + \frac{k_{20}\,k_{10}}{3\pi B^3}\Bigg[B^3(l^3-x^3)\cosh^{-1}\frac{1-B^2lx}{B(l-x)}+B^3(l^2+x^3)\cosh^{-1}\frac{1+B^2lx}{B(l+x)}+ \\
& + \left(2\sin^{-1}Bl\,B^2x^2+\sin^{-1}Bl-Bl\sqrt{1-B^2l^2}\right)\sqrt{1-B^2x^2}\Bigg]+ \\
& + x\Bigg\{\frac{k_{21}}{2\pi B^2}\Bigg[B^2(l^2-x^2)\left(\cosh^{-1}\frac{1-B^2lx}{B(l-x)}-\cosh^{-1}\frac{1+B^2lx}{B(l+x)}\right)+ \\
& + 2\sin^{-1}Bl\,Bx\sqrt{1-B^2x^2}\Bigg]+ \\
& + \frac{k_{21}k_{11}}{3\pi B^3}\Bigg[B^3(l^3-x^3)\cosh^{-1}\frac{1-B^2lx}{B(l-x)}-B^3(l^3+x^3)\cosh^{-1}\frac{1+B^2lx}{B(l+x)}+ \\
& + 2B^3x^3\cosh^{-1}\frac{1}{Bx}+2\left(1-\sqrt{1-B^2l^2}\right)Bx\sqrt{1-B^2x^2}\Bigg]\Bigg\}-L.
\end{aligned} \tag{22}$$

Through the superpositioning of the three wing components, the resulting imaginary wing is obtained, equivalent from an aerodynamic point of view with the real wing, for which the axis of disturbance velocity will be

$$\mathcal{U}_1 = \mathcal{U}_{1t} + \mathcal{U}_{1i} + \mathcal{U}_{1e}. \tag{23}$$

We observe that the velocity U_{1l} on the higher side of the wing is equal and of opposed sign with that of the lower side, corresponding to a thin wing.

THE SIMPLIFIED CASE OF CONCENTRATED SOURCES

If we will consider, for simplification, a brief variation of vertical velocities in the abscissa point $s' = c$ due to the influence of the vortex from the right of this point, we will have the case of a brief drop of incidence, corresponding to an "edge" of separation of vertical velocities $(w_{10}^{(0)} + w_{01}^{(0)} c)\, x_1$ and $(w_{10}^{(1)} + w_{01}^{(1)} c)\, x_1$ [2]. The brief drop in the distribution of incidences will correspond to a variation of vertical velocities expressed analytically by the concentrated source in $s'=c$ of intensities $Q_0 = Q_0^{(n)} + Q_0^{(f)}$, for the straight part of the wing, where $Q_0^{(n)}$ corresponds to natural symmetries, and $Q_0^{(f)}$ to forced incidences.

Following in continuation the path presented above, as in (1) and (3), we will obtain for the three wing components the following expressions of the axis of disturbance velocities;

$$\mathcal{U}_{1l} = \frac{1}{x_1}\mathcal{U}_l = \frac{A_{20} + A_{22}x^2}{\sqrt{l^2 - x^2}} + \frac{2}{\pi}(Q_{20} + Q_{21}x)\cosh^{-1}\sqrt{\frac{(l+c)(l-x)}{2l(c-x)}} +$$
$$+ \frac{2}{\pi}(Q_{20} - Q_{21}x)\cosh^{-1}\sqrt{\frac{(l+c)(l+x)}{2l(c+x)}} =$$
$$= \frac{A_{20} + A_{22}x^2}{\sqrt{l^2 - x^2}} + \frac{2}{\pi}\left[Q_{20}\cosh^{-1}\sqrt{\frac{l^2 - x^2}{c^2 - x^2}} \pm \right.$$
$$\left. \pm\, Q_{21}\, x \cosh^{-1}\frac{c}{l}\sqrt{\frac{l^2 - x^2}{c^2 - x^2}}\right] \tag{24}$$

for the "lift wing ",

$$\mathcal{U}_{1s} = \frac{1}{x_1}\mathcal{U}_s = \frac{2}{\pi}(Q_{20} + Q_{21}x)\cosh^{-1}\sqrt{\frac{(1 + Bc)(1 - Bx)}{2B(c - x)}} +$$

$$+\frac{2}{\pi}(Q_{20}-Q_{21}x)\cosh^{-1}\sqrt{\frac{(1+Bc)(1+Bx)}{2B(c+x)}}+L=$$

$$=\frac{2}{\pi}\left[Q_{20}\cosh^{-1}\sqrt{\frac{1-B^2x^2}{B^2(c^2-x^2)}}\pm Q_{21}\,x\cosh^{-1}Bc\sqrt{\frac{1-B^2x^2}{B^2(c^2-x^2)}}\right]+L \tag{25}$$

for the "wing of symmetrical thickness", having the slope equal with the incidence $\frac{\alpha'}{\chi_1}$ of the lift wing, which was deduced from (17) considering the sources of intensities Q_{20} and Q_{21}, concentrated in $y = c$ in place of liniar distributions, (2), and

$$\mathcal{U}'_{1c}\equiv\mathcal{U}_{1c} \tag{26}$$

for the "compensating wing" of slope, where U_{1c} has the expression given by (22).

We will moreover remark that U'_{1c} given as the sign of the axes velocities is different on the two sides of the wing. The expression of the total axis vleocities will be that given by (23), which will give us the distribution of pressure on the deformed wing.

3. THE DETERMINATION OF THE CONSTANTS

For the determination of the constants q^*_{20} and q^*_{21}, which appear in the expression of U_{11}(16), will begin from the conditions

$$\mathrm{Im}\oint_{\sigma}\sqrt{1-B^2x^2}\,\frac{d^2}{dx^2}\left(\frac{d\mathcal{U}_{11}}{d\eta}\right)dx=\frac{dw'_{10}}{d\eta}, \tag{27a}$$

$$-\mathrm{Im}\oint_{\sigma}\frac{\sqrt{1-B^2x^2}}{x}\,\frac{d^2}{dx^2}\left(\frac{d\mathcal{U}_{11}}{d\eta}\right)dx=\frac{dw'_{01}}{d\eta}, \tag{27b}$$

deduced from the theory of conical motion (4), the integration being accomplished on a semicircle σ of very small radius around

a certain point y = n on the wing, contained in the interval (y = s, y = 1) (fig. 2).

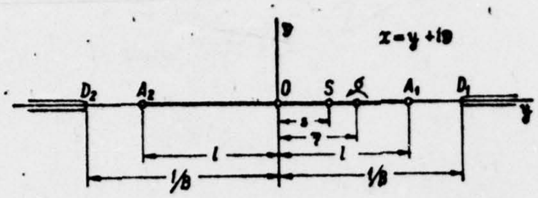

Fig. 2

Proceeding in this way, we will obtain the relations

$$q'_{20} = \frac{\eta}{(1-B^2\eta^2)^{3/2}}\left[(2-B^2\eta^2)\frac{dw'_{10}}{d\eta} + \eta\frac{dw'_{01}}{d\eta}\right], \qquad (28a)$$

$$q'_{21} = -\frac{\eta}{(1-B^2\eta^2)^{3/2}}\left[\frac{dw'_{10}}{d\eta} + B^2\eta^3\frac{dw'_{01}}{d\eta}\right], \qquad (28b)$$

which establishes the dependence from the variation of sources and the distribution of vertical velocities on the thin imaginary wing.

Starting from these relations and keeping in mind (14), we place the conditions at the limit in the points n = s and n = 1, for the vertical velocity $w' = w'_{10}x_1 + w'_{01}x_2$, and we will obtain

$$w_{10}^{(1)} - w_{10}^{(0)} = \frac{q_{20}^{*(n)}}{l}\int_s^l \frac{(l-\eta)(1-B^2\eta^2)^{3/2}}{\eta(2-B^2\eta^2)}\,d\eta, \qquad (29a)$$

$$w_{10}^{(1)} - w_{10}^{(0)} = -\frac{q_{21}^{*(n)}}{l}\int_s^l (l-\eta)(1-B^2\eta^2)^{3/2}\,d\eta, \qquad (29b)$$

in the case of natural symmetries, for which $q'_{20} = q'^{(n)}_{20}$, and $w'_{01} = 0$. Here we find the first relations among the constants

$q_{20}^{*(n)}$, $q_{21}^{*(n)}$, $w_{10}^{(0)}$ şi $w_{10}^{(1)}$: :

$$\frac{1}{2}\, q_{20}^{*(n)} \Bigg[\sqrt{1-B^2 l^2} - \sqrt{1-B^2 s^2} - \cosh^{-1}\frac{1}{Bl} + $$

$$+ \cosh^{-1}\frac{1}{Bs} - \left(1-\frac{s}{l}\right)\sqrt{1-B^2 s^2} + \frac{1}{Bl}\left(\sin^{-1} Bl - \sin^{-1} Bs\right) -$$

$$- \frac{1}{2}\left(\sin^{-1}\frac{1-Bl\sqrt{2}}{\sqrt{2}-Bl} + \sin^{-1}\frac{1+Bl\sqrt{2}}{\sqrt{2}+Bl} - \sin^{-1}\frac{1-Bs\sqrt{2}}{\sqrt{2}-Bs} - \right.$$

$$\left. - \sin^{-1}\frac{1+Bs\sqrt{2}}{\sqrt{2}+Bs}\right) + \frac{1}{\sqrt{2}\,Bl}\left(\sin^{-1}\frac{1-Bl\sqrt{2}}{\sqrt{2}-Bl} - \sin^{-1}\frac{1+Bl\sqrt{2}}{\sqrt{2}+Bl} - \right.$$

$$\left. - \sin^{-1}\frac{1-Bs\sqrt{2}}{\sqrt{2}-Bs} + \sin^{-1}\frac{1+Bs\sqrt{2}}{\sqrt{2}+Bs}\right)\Bigg] = w_{10}^{(1)} - w_{10}^{(0)}, \qquad (30a)$$

$$- \frac{q_{21}^{*(n)}}{8B}\Bigg[3\left(\sin^{-1} Bl - \sin^{-1} Bs + Bl\sqrt{1-B^2 l^2} - Bs\sqrt{1-B^2 s^2}\right) +$$

$$+ 2\left(Bl(1-B^2 l^2)^{3/2} - Bs(1-B^2 s^2)^{3/2}\right) + \frac{8}{5\,Bl}\Big((1-B^2 l^2)^{5/2} -$$

$$- (1-B^2 s^2)^{5/2}\Big)\Bigg] = w_{10}^{(1)} - w_{10}^{(0)}. \qquad (30b)$$

For the incidences with forced symmetry, making $q'_{20} = q'^{(f)}_{20}$ and $w'_{10} = 0$, from (28a) and (28b) we obtain

$$w_{01}^{(1)} - w_{01}^{(0)} = \frac{q_{20}^{*(f)}}{l}\int_s^l \frac{(l-\eta)\,(1-B^2\eta^2)^{3/2}}{\eta^2}\,d\eta, \qquad (31a)$$

$$w_{01}^{(1)} - w_{01}^{(0)} = -\frac{q_{21}^{*(f)}}{B^2 l}\int_s^l \frac{(l-\eta)\,(1-B^2\eta^2)^{3/2}}{\eta^3}\,d\eta, \qquad (31b)$$

from which we deduce the following **relations:**

$$\frac{q_{20}^{*(f)}}{l}\Bigg\{\left(1+\frac{l}{s}\right)\sqrt{1-B^2 s^2} - 2\sqrt{1-B^2 l^2} - \frac{3}{2}\,Bl\left(\sin^{-1} Bl - \sin^{-1} Bs\right) -$$

$$- \frac{1}{2}\,B^2 l^2\left(\sqrt{1-B^2 l^2} - \frac{s}{l}\sqrt{1-B^2 s^2}\right) - \frac{1}{3}\left[(1-B^2 l^2)^{3/2} -\right.$$

$$\left. - (1-B^2 s^2)^{3/2}\right] + \cosh^{-1}\frac{1}{Bl} - \cosh^{-1}\frac{1}{Bs}\Bigg\} = w_{01}^{(1)} - w_{01}^{(0)}, \qquad (32a)$$

$$\frac{q^{*(f)}_{21}}{2B^2 l^2}\left\{(1 - B^2 l^2)^{3/2} + \left[B^2 l^2\left(2 - \frac{s}{l}\right) + \frac{l^2}{s^2}\left(1 - 2\,\frac{s}{l}\right)\right]\sqrt{1 - B^2 s^2} + \right.$$

$$\left. + 3\,Bl\left[Bl\left(\cosh^{-1}\frac{1}{Bl} - \cos^{-1}\frac{1}{Bs}\right) + \sin^{-1} Bl - \sin^{-1} Bs\right]\right\} =$$

$$= w^{(0)}_{01} - w^{(1)}_{01}. \tag{32b}$$

On the other ~~part~~ hand, the mean vertical velocity or the incidence of the real wing, given by (3), equal with that of the first wing components as with that of the resulting imaginary wings, is obtained starting from relation(2). In this way, keeping in mind that the two thick wings 2). and 3). are compensated reciprocally, the relation (2) is written simply

$$\frac{1}{l}\int_0^l (w'_{10} + w'_{01}\,\eta)\,\mathrm{d}\eta = w_{10} + \frac{1}{2}\,w_{01}\,l, \tag{33}$$

corresponding only to thin wing component 1).

Next, the relation can be put in the form

$$w^{(0)}_{10}\,s + \frac{1}{2}\,w^{(0)}_{01}\,s^2 + \int_s^l (w'_{10} + w'_{01}\,\eta)\,\mathrm{d}\eta = w_{10}\,l + \frac{1}{2}\,w_{01}\,l^2, \tag{34}$$

which, after accomplishing the integral, becomes

$$w_{10} - w^{(1)}_{10} = \frac{q^{*(n)}_{21}}{48B^3 l^2}\left\{\frac{8}{5}\left[B(6l - 5\,s)\,(1 - B^2 s^2)^{5/2} - Bl(1 - B^2 l^2)^{5/2}\right] - \right.$$

$$- 2\left[Bl(1 - B^2 l^2)^{3/2} - Bs\,(1 - B^2 s^2)^{3/2}\right] -$$

$$\left. - 3\left(Bl\sqrt{1 - B^2 l^2} - Bs\sqrt{1 - B^2 s^2} - \sin^{-1} Bl + \sin^{-1} Bs\right)\right\}, \tag{35}$$

in the case of natural symmetries of the incidences and

$$w^{(1)}_{01} - w_{01} = \frac{q^{*(f)}_{20}}{8\,Bl^2}\left\{3\left[\sin^{-1} Bl - \sin^{-1} Bs + Bl\sqrt{1 - B^2 l^2} - Bs\sqrt{1 - B^2 s^2}\right] + \right.$$

$$+ 2\left[Bl(1 - B^2 l^2)^{3/2} - Bs\,(1 - B^2 s^2)^{3/2}\right] + \frac{8}{5\,Bl}\left[(1 - B^2 l^2)^{5/2} - \right.$$

$$\left.\left. - (1 - B^2 s^2)^{5/2}\right]\right\}, \tag{36}$$

for the case of incidence with forced symmetry.

For the determination of the constants A_{20} and A_{22} which appear in expression (16), we will take into consideration the variation of vertical vleocities $\overline{w'}_{x1} = w'_{10} + w'_{01}y$, on the first wing component, from a point on the wing to one of nought vertical velocity (for example on Mach cone), as is preceeded in the theory of conical motion. As in (1) and (3), in order to avoid some difficult calculations to determine these constants, we will consider, through approximation, that the sources distributed in a line in the interval (s, l) are concentrated in $y = s'$ of intensities Q_{20} and Q_{21}, such that we have

$$s' = s + \frac{l - s}{3}, \quad Q_{20} = \frac{1}{2} \dot{q}_{20}\, l \left(1 - \frac{s}{l}\right)^2, \quad Q_{21} = \frac{1}{2} \dot{q}_{21}\, l \left(1 - \frac{s}{l}\right)^2. \tag{37}$$

Proceeding in this way, we can write the relations

$$-\operatorname{Im} \int_{\text{aripă}}^{\text{cercul Mach}} \frac{\sqrt{1 - B^2 x^2}}{x} \frac{d^2 \mathcal{U}'_{11}}{dx^2} dx = w_{01}^{(0)}, \tag{38a}$$

$$\operatorname{Im} \int_{\text{aripă } (l)}^{\text{cercul Mach } (1/B)} \sqrt{1 - B^2 x^2} \frac{d^2 \mathcal{U}'_{11}}{dx^2} dx = w_{10}^{(1)}, \tag{38b}$$

where U'_{11} represents the axis of disturbance velocity for the simplified case of concentrated sources in $y = s'$, given by expression (24).

Through the accomplishment of the integrals which appear above, on a circle of very small radius around the origin for (38a) and on the axis of the abscissa ($\mathfrak{z} = 0$), between the limits l and $1/B$ for (38b), results

$$A_{20} + 2l^2 A_{22} + \frac{2}{\pi} l^2 \left\{ q_{20}^* \left[\left(1 + \frac{s}{l}\right) \sqrt{1 - \frac{s^2}{l^2}} - \cos^{-1}\frac{s}{l} - \cosh^{-1}\frac{l}{s} \right] + \right.$$
$$\left. + q_{21}^* l \left[2\left(\cosh^{-1}\frac{l}{s} - \sqrt{1 - \frac{s^2}{l^2}}\right) + \frac{s}{l}\sqrt{1 - \frac{s^2}{l^2}} - \cos^{-1}\frac{s}{l} \right] \right\} =$$
$$= \frac{2}{\pi} w_{01}^{(0)} l^3, \tag{39a}$$

$$(A_{20} + A_{22} l^2) \frac{B^2}{1 - B^2 l^2} [E(k) - K(k)] + A_{22} E(k) -$$
$$- \frac{2}{\pi}\left(1 - \frac{s}{l}\right)^{5/2} \sqrt{2 + \frac{s}{l}} \left(\frac{1}{3} B^2 l^2 Y q_{20}^* + \frac{Y - K(k)}{l + 2s} q_{21}^* l^2\right) = w_{10}^{(1)}, \tag{39b}$$

in which we noted

$$Y = \frac{\pi}{2} \frac{s'}{\sqrt{(l^2 - s'^2)(1 - B^2 s'^2)}} - \frac{B^2 s'^2}{1 - B^2 s'^2} \Pi(\rho, k) + \frac{1}{1 - B^2 s'^2} K(k), \tag{40a}$$

$$\Pi(\rho, k) = K(k) + \frac{\sqrt{1 - B^2 s'^2}}{B^2 s' \sqrt{l^2 - s'^2}} \left[\frac{\pi}{2} - K(k) E(\varphi_0, k') + \right.$$
$$\left. + (K(k) - E(k)) F(\varphi_0, k')\right], \tag{40b}$$

and $K(k), E(k)$ and $\Pi(\rho, k)$ represent the completed eliptical integrals of the first, second and third instances respectively, having the module k and parameter p given by the relations

$$k = \sqrt{1 - B^2 l^2},\ \rho = B^2 s'^2 - 1 = \frac{1}{9} B^2 (l + 2s)^2 - 1,\ k' = Bl,\ \varphi_0 =$$
$$= \sin^{-1}\frac{s'}{l}. \tag{41}$$

Due to the separation of flow at the edges of the wing and resulting vortexes on the higher side of the wing, finite velocities in those points are realized. Imposing these conditions we will be able to write the relation

$$A_{20} + l^2 A_{22} = 0. \tag{42}$$

Next , it is observed that inthe expression of $\frac{1}{x_1} U_c$ given by (22) appears the constants k_{10}, k_{20}, k_{11} and k_{21}, which follow to be determined. Taking into consideration the roll of the third wing component, which will have the mean slope $-w_{10}x_1 \pm w_{01}x_2$, and keeping in mind relations (20a) and (20b), we can write, similar with (33):

$$\frac{1}{l}\int_0^l (w_{10}'' + w_{01}''\eta)\, d\eta = -w_{10} - \frac{1}{2} w_{01} l, \tag{43}$$

from where we deduce the relations

$$(w_{10} - w_{10}^{(1)})\, l = -k_{21}^{(n)} \int_0^l \eta^2 (1 + k_{11}^{(n)} \eta)(1 - B^2\eta^2)^{3/2}\, d\eta, \tag{44a}$$

$$(w_{10} - w_{10}^{(1)})\, l = k_{20}^{(n)} \int_0^l \eta^2 (1 + k_{10}^{(n)} \eta) \frac{(1 - B^2\eta^2)^{3/2}}{\eta(2 - B^2\eta^2)}\, d\eta, \tag{44b}$$

in the case of incidence with natural symmetry, which, in the course of accomplishing the integrals, becomes

$$\begin{aligned} w_{10} - w_{10}^{(1)} = \frac{k_{21}^{(n)}}{2B^2} \Bigg\{ & \frac{1}{3}(1 - B^2 l^2)^{5/2} - \frac{1}{12}(1 - B^2 l^2)^{3/2} - \\ & - \frac{1}{8}\left(\sqrt{1 - B^2 l^2} + \frac{\sin^{-1} Bl}{Bl}\right) - \frac{2k_{11}^{(n)}}{B^2 l}\Big[\frac{1}{5}(1 - (1 - B^2 l^2)^{5/2}) - \\ & - \frac{1}{7}(1 - (1 - B^2 l^2)^{7/2})\Big]\Bigg\}, \end{aligned} \tag{45a}$$

$$\begin{aligned} w_{10} - w_{10}^{(1)} = \frac{k_{20}^{(n)}}{8B^2 l} \Bigg\{ & \frac{k_{10}^{(n)}}{B}\Big[Bl\sqrt{1 - B^2 l^2}(3 + 2B^2 l^2) - 11 \sin^{-1} Bl - \\ & - \frac{8}{\sqrt{2}}\left(\sin^{-1}\frac{1 - Bl\sqrt{2}}{\sqrt{2} - Bl} - \sin^{-1}\frac{1 + Bl\sqrt{2}}{\sqrt{2} + Bl}\right)\Big] + \\ & + \frac{8}{3}\left[B^2 l^2 \sqrt{1 - B^2 l^2} - 2(1 - \sqrt{1 - B^2 l^2})\right] + \\ & + 2\pi - 4\left(\sin^{-1}\frac{1 - Bl\sqrt{2}}{\sqrt{2} - Bl} + \sin^{-1}\frac{1 + Bl\sqrt{2}}{\sqrt{2} + Bl}\right)\Bigg\}. \end{aligned} \tag{45b}$$

For the particular case of forced symmetry, we will start similarly from the relation (42), keeping in mind and wellknown from (20a), (20b), (28a) and (28b), and obtain

$$(w_{01} - w_{01}^{(1)})\, l^2 = k_{20}^{(f)} \int_0^l \eta\, (1 + k_{10}^{(f)} \eta)\, (1 - B^2 \eta^2)^{3/2}\, d\eta, \tag{46a}$$

$$(w_{01} - w_{01}^{(1)})\, l^2 = -\frac{k_{21}^{(f)}}{B^2} \int_0^l (1 + k_{10}^{(f)} \eta)\, (1 - B^2 \eta^2)^{3/2}\, d\eta, \tag{46b}$$

which, after accomplishing the calculations, terminates in the form

$$w_{01}^{(1)} - w_{01} = \frac{k_{20}^{(f)}}{B^2 l^2} \left\{ \frac{1}{5} \left((1 - B^2 l^2)^{5/2} - 1 \right) + \frac{1}{2} k_{10}^{(f)} l \left[\frac{1}{3} (1 - B^2 l^2)^{5/2} - \right.\right.$$
$$\left.\left. - \frac{1}{12} (1 - B^2 l^2)^{3/2} - \frac{1}{8} \left(\sqrt{1 - B^2 l^2} + \frac{\sin^{-1} Bl}{Bl} \right) \right] \right\}. \tag{47a}$$

$$w_{01}^{(1)} - w_{01} = \frac{k_{21}^{(f)}}{B^3 l^2} \left\{ \frac{1}{8} Bl \left[2(1 - B^2 l^2)^{3/2} + 3 \left(\sqrt{1 - B^2 l^2} + \frac{\sin^{-1} Bl}{Bl} \right) \right] - \right.$$
$$\left. - \frac{k_{11}^{(f)}}{5B} \left((1 - B^2 l^2)^{5/2} - 1 \right) \right\}. \tag{47b}$$

We will remark, in continuation, that equations (30a), (30b), (35), (39a), (39b) and (42) constitute the system for the determination of the constants for the curved thin delta wing (with natural symmetry). Thus, in (39a), in which we replaced q^* for $q^{*(n)}$ and $w_{01}^{(o)} = 0$, and in (42) results

$$A_{20}^{(n)} = \frac{2}{\pi} l^2 \left\{ q_{20}^{*(n)} \left[\left(1 + \frac{s}{l} \right) \sqrt{1 - \frac{s^2}{l^2}} - \cos^{-1} \frac{s}{l} - \cosh^{-1} \frac{l}{s} \right] + \right.$$
$$\left. + q_{21}^{*(n)} l \left[2 \left(\cosh^{-1} \frac{l}{s} - \sqrt{1 - \frac{s^2}{l^2}} \right) + \frac{s}{l} \sqrt{1 - \frac{s^2}{l^2}} - \cosh^{-1} \frac{s}{l} \right] \right\}, \tag{48a}$$

$$A_{22}^{(n)} = -\frac{1}{l^2} A_{20}^{(n)}, \tag{48b}$$

and from (35 and (39b) we reduce the relation

$$\frac{2}{\pi} q_{20}^{*(n)} \left\{ \left[\left(1 + \frac{l}{s} \right) \sqrt{1 - \frac{s^2}{l^2}} - \cos^{-1} \frac{s}{l} - \cosh^{-1} \frac{l}{s} \right] E(k) + \right.$$
$$\left. + \frac{1}{3} B^2 l^2 Y \left(1 - \frac{s}{l} \right)^{5/2} \sqrt{2 + \frac{s}{l}} \right\} +$$

$$+\frac{2}{\pi}q_{21}^{*(n)}l\left\{\left[2\left(\cosh^{-1}\frac{l}{s}-\sqrt{1-\frac{s^2}{l^2}}\right)+\frac{s}{l}\sqrt{1-\frac{s^2}{l^2}}-\cos^{-1}\frac{s}{l}\right]E(k)+\right.$$

$$+\left(1-\frac{s}{l}\right)^{5/2}\sqrt{2+\frac{s}{l}}\,\frac{Y-K(k)}{1+2\frac{s}{l}}-$$

$$-\frac{\pi}{96B^3l^3}\left[\frac{8}{5}\left(B(6l-5s)(1-B^2s^2)^{5/2}-Bl(1-B^2l^2)^{5/2}\right)-\right.$$

$$-2\left(Bl(1-B^2l^2)^{3/2}-Bs(1-B^2s^2)^{3/2}\right)-$$

$$\left.\left.-3\left(Bl\sqrt{1-B^2l^2}-Bs\sqrt{1-B^2s^2}-\sin^{-1}Bl+\sin^{-1}Bs\right)\right]\right\}=-w_{10}; \quad (49)$$

then, together with the relation

$$4Bq_{20}^{*(n)}\left\{\sqrt{1-B^2l^2}-\sqrt{1-B^2s^2}-\cosh^{-1}\frac{1}{Bl}+\cosh^{-1}\frac{1}{Bs}-\right.$$

$$-\left(1-\frac{s}{l}\right)\sqrt{1-B^2s^2}+\frac{1}{Bl}(\sin^{-1}Bl-\sin^{-1}Bs)-$$

$$-\frac{1}{2}\left[\sin^{-1}\frac{1-Bl\sqrt{2}}{\sqrt{2}-Bl}+\sin^{-1}\frac{1+Bl\sqrt{2}}{\sqrt{2}+Bl}-\sin^{-1}\frac{1-Bs\sqrt{2}}{\sqrt{2}-Bs}-\right.$$

$$\left.-\sin^{-1}\frac{1+Bs\sqrt{2}}{\sqrt{2}+Bs}\right]+\frac{1}{\sqrt{2}Bl}\left[\sin^{-1}\frac{1-Bl\sqrt{2}}{\sqrt{2}-Bl}-\sin^{-1}\frac{1+Bl\sqrt{2}}{\sqrt{2}+Bl}-\right.$$

$$\left.\left.-\sin^{-1}\frac{1-Bs\sqrt{2}}{\sqrt{2}-Bs}+\sin^{-1}\frac{1+Bs\sqrt{2}}{\sqrt{2}+Bs}\right]\right\}+$$

$$+q_{21}^{*(n)}\left\{3\left[\sin^{-1}Bl-\sin^{-1}Bs+Bl\sqrt{1-B^2l^2}-Bs\sqrt{1-B^2s^2}\right]+\right.$$

$$+2[Bl(1-B^2l^2)^{3/2}-Bs(1-B^2s^2)^{3/2}]+$$

$$\left.+\frac{8}{5Bl}[(1-B^2l^2)^{5/2}-(1-B^2s^2)^{5/2}]\right\}=0, \quad (50)$$

obtained in (30a) and (30b), is formed the system of the two equations which determine the constants $q_{20}^{*(n)}$ AND $q_{21}^{*(n)}$.

Proceeding in this way, we will find

$$q_{20}^{*(n)}=-\frac{I_{21}^{(n)}}{I_{21}^{(n)}J_{20}^{(n)}-I_{20}^{(n)}J_{21}^{(n)}}\,w_{10}, \quad (51)$$

$$q_{21}^{*(n)}=\frac{I_{20}^{(n)}}{I_{21}^{(n)}J_{21}^{(n)}-I_{20}^{(n)}J_{21}^{(n)}}\,w_{10}, \quad (51)$$

where we made the notations

$$I_{20}^{(n)} = 2B\left\{2\left[\sqrt{(1-B^2l^2}-\left(2-\frac{s}{l}\right)\sqrt{1-B^2s^2}-\cosh^{-1}\frac{1}{Bl}+\cosh^{-1}\frac{1}{Bs}\right]\cdot\right.$$
$$+\frac{1}{Bl}\left[2(\sin^{-1}Bl-\sin^{-1}Bs)-(Bl-\sqrt{2})\left(\sin^{-1}\frac{1-Bl\sqrt{2}}{\sqrt{2}-Bl}-\sin^{-1}\frac{1-Bs\sqrt{2}}{\sqrt{2}-Bs}\right)\cdot\right.$$
$$\left.\left.-(Bl+\sqrt{2})\left(\sin^{-1}\frac{1-Bl\sqrt{2}}{\sqrt{2}-Bl}-\sin^{-1}\frac{1-Bs\sqrt{2}}{\sqrt{2}-Bs}\right)\right]\right\}, \qquad \text{(52}$$

$$I_{21}^{(n)} = 3(\sin^{-1}Bl-\sin^{-1}Bs+Bl\sqrt{1-B^2l^2}-Bs\sqrt{1-B^2s^2})+$$
$$+2[Bl(1-B^2l^2)^{3/2}-Bs(1-B^2s^2)^{3/2}]+$$
$$+\frac{8}{5Bl}[(1-B^2l^2)^{5/2}-(1-B^2s^2)^{5/2}], \qquad \text{(52}$$

$$J_{20}^{(n)} = \frac{2}{\pi}\left\{\left[\left(1+\frac{l}{s}\right)\sqrt{1-\frac{s^2}{l^2}}-\cos^{-1}\frac{s}{l}-\cosh^{-1}\frac{l}{s}\right]E(k)+\right.$$
$$\left.+\frac{1}{3}B^2l^2Y\left(1-\frac{s}{l}\right)^{5/2}\sqrt{2+\frac{s}{l}}\right\}. \qquad \text{(52c)}$$

$$J_{21}^{(n)} = \frac{2}{\pi}l\left\{\left[2\left(\cosh^{-1}\frac{l}{s}-\sqrt{1-\frac{s^2}{l^2}}\right)+\frac{s}{l}\sqrt{1-\frac{s^2}{l^2}}-\cos^{-1}\frac{s}{l}\right]E(k)+\right.$$
$$+\left(1-\frac{s}{l}\right)^{5/2}\sqrt{2+\frac{s}{l}}\,\frac{Y-K(k)}{1+2\frac{s}{l}}-$$
$$-\frac{\pi}{96B^3l^3}\left[\frac{8}{5}(B(6l-5s)(1-B^2s^2)^{5/2}-Bl(1-B^2l^2)^{5/2})-\right.$$
$$-2(Bl(1-B^2l^2)^{3/2}-Bs(1-B^2s^2)^{3/2})-$$
$$\left.\left.-3(Bl\sqrt{1-B^2l^2}-Bs\sqrt{1-B^2s^2}-\sin^{-1}Bl+\sin^{-1}Bs)\right]\right\}. \qquad \text{(52d)}$$

From (35), (45a) and (45b), considering for k_{10} and k_{11} suitable values,

$$k_{10} = k_{11} = -\frac{1}{2l}, \qquad \text{(53)}$$

results, for the constants $k_{20}^{(n)}$ and $k_{21}^{(n)}$, the expressions:

$$k_{20}^{(n)} = \frac{q_{21}^{*(n)}}{6Bl}\left\{\frac{8}{5}[B(6l-5s)(1-B^2s^2)^{5/2}-Bl(1-B^2l^2)^{5/2}]-\right.$$
$$-2[Bl(1-B^2l^2)^{3/2}-Bs(1-B^2s^2)^{3/2}]-$$

$$
-3\left(Bl\sqrt{1-B^2l^2}-Bs\sqrt{1-B^2s^2}-\sin^{-1}Bl+\sin^{-1}Bs\right)\Big\}\times
$$

$$
\times\Big\{\frac{8}{3}\Big[B^2l^2\sqrt{1-B^2l^2}-2\left(1-\sqrt{1-B^2l^2}\right)+
$$

$$
+2\pi-4\left(\sin^{-1}\frac{1-Bl\sqrt{2}}{\sqrt{2}-Bl}+\sin^{-1}\frac{1+Bl\sqrt{2}}{\sqrt{2}+Bl}\right)\Big]-
$$

$$
-\frac{1}{2\,Bl}\left[Bl\sqrt{1-B^2l^2}\,(3+2B^2l^2)-11\sin^{-1}Bl\right]-
$$

$$
-4\sqrt{2}\left(\sin^{-1}\frac{1-Bl\sqrt{2}}{\sqrt{2}-Bl}-\sin^{-1}\frac{1+Bl\sqrt{2}}{\sqrt{2}+Bl}\right)\Big\}^{-1}, \tag{54a}
$$

$$
k_{21}^{(n)}=\frac{q_{21}^{*(n)}}{24\,Bl^2}\Big\{\frac{8}{5}\left[B(6l-5s)\,(1-B^2s^2)^{5/2}-Bl(1-B^2l^2)^{5/2}\right]-
$$

$$
-2\left[Bl(1-B^2l^2)^{3/2}-Bs(1-B^2s^2)^{3/2}\right]-
$$

$$
-3\left(Bl\sqrt{1-B^2l^2}-Bs\sqrt{1-B^2s^2}-\sin^{-1}Bl+\sin^{-1}Bs\right)\Big\}\times
$$

$$
\times\Big\{\frac{1}{3}(1-B^2l^2)^{5/2}-\frac{1}{12}(1-B^2l^2)^{3/2}-\frac{1}{8}\left(\frac{1}{Bl}\sin^{-1}Bl+\sqrt{1-B^2l^2}\right)+
$$

$$
+\frac{1}{B^2l^2}\Big[\frac{1}{5}\left(1-(1-B^2l^2)^{5/2}\right)-\frac{1}{7}\left(1-(1-B^2l^2)^{7/2}\right)\Big\}^{-1}. \tag{54b}
$$

Considering now the torsioned wing, having a forced symmetry of incidences in reference with plane Ox_1x_3 (fig. 1), the system which will determine the constants will be formed from the equations (32a), (32b), (36), (39a), (39b) and (42), in which $q^* = q^{*(f)}$ and $w_{10}^{(i)} = o$. We will obtain, successively, from (39b) and from (42):

$$
A_{20}^{(f)}=-\frac{2l^2}{\pi E(k)}\left(1-\frac{s}{l}\right)^{5/2}\sqrt{2+\frac{s}{l}}\left(\frac{1}{3}B^2l^2Yq_{20}^{*(f)}+\frac{Y-K(k)}{1+2\frac{s}{l}}q_{21}^{*(f)}l\right), \tag{55a}
$$

$$
A_{22}^{(f)}=-\frac{A_{20}^{(f)}}{l^2}, \tag{55b}
$$

and from the other equations result the values of the constants $q_{20}^{*(F)}$ AND $q_{21}^{*(F)}$ similar with (51a) and (51b), in which it is noted

$$I_{20}^{(f)} = 2B^2 l \left\{ \left(1+\frac{l}{s}\right)\sqrt{1-B^2s^2} - 2\sqrt{1-B^2l^2} + \cosh^{-1}\frac{1}{Bl} - \cosh^{-1}\frac{1}{Bs} - \right.$$

$$-\frac{1}{2}Bl\left[Bl\left(\sqrt{1-B^2l^2}-\frac{s}{l}\sqrt{1-B^2s^2}\right)+3(\sin^{-1}Bl-\sin^{-1}Bs)\right]-$$

$$\left. -\frac{1}{3}\left[(1-B^2l^2)^{3/2}-(1-B^2s^2)^{3/2}\right]\right\}, \qquad (56a)$$

$$I_{21}^{(f)} = (1-B^2l^2)^{3/2} + \frac{l^2}{s^2}\left[1-2\frac{s}{l}+B^2s^2\left(2-\frac{s}{l}\right)\right]\sqrt{1-B^2s^2} +$$

$$+3Bl\left[\sin^{-1}Bl-\sin^{-1}Bs+Bl\left(\cosh^{-1}\frac{1}{Bl}-\cosh^{-1}\frac{1}{Bs}\right)\right]. \qquad (56b)$$

$$J_{20}^{(f)} = \frac{1}{Bl^2}\left\{\frac{3}{8}\left[\sin^{-1}Bl-\sin^{-1}Bs+Bl\sqrt{1-B^2l^2}-Bs\sqrt{1-B^2s^2}\right]+\right.$$

$$+\frac{1}{4}\left[Bl(1-B^2l^2)^{3/2}-Bs(1-B^2s^2)^{3/2}\right]+$$

$$+\frac{1}{5Bl}\left[(1-B^2l^2)^{5/2}-(1-B^2s^2)^{5/2}\right]+$$

$$+Bl\left[\cos^{-1}\frac{s}{l}+\cosh^{-1}\frac{l}{s}-\left(1+\frac{l}{s}\right)\sqrt{1-\frac{s^2}{l^2}}-\right.$$

$$\left.\left.-\frac{B^2l^2Y}{3E(k)}\left(1-\frac{s}{l}\right)^{5/2}\sqrt{2+\frac{s}{l}}\right]\right\}, \qquad (56c)$$

$$J_{21}^{(f)} = \frac{1}{2B^2l^2}\left\{(1-B^2l^2)^{3/2}+\frac{l^2}{s^2}\left[1-2\frac{s}{l}+B^2s^2\left(2-\frac{s}{l}\right)\right]\sqrt{1-B^2s^2}+\right.$$

$$\left.+3Bl\left[\sin^{-1}Bl-\sin^{-1}Bs+Bl\left(\cosh^{-1}\frac{1}{Bl}-\cosh^{-1}\frac{1}{Bs}\right)\right]\right\}+$$

$$+\cos^{-1}\frac{s}{l}-2\cosh^{-1}\frac{l}{s}+\left(2-\frac{s}{l}\right)\sqrt{1-\frac{s^2}{l^2}}-$$

$$-\frac{Y-K(k)}{\left(1+2\frac{s}{l}\right)E(k)}\left(1-\frac{s}{l}\right)^{5/2}\sqrt{2+\frac{s}{l}}. \qquad (56d)$$

The constants $k_{20}^{(f)}$ and $k_{21}^{(f)}$ are deduced from (47a) and (47b), keeping in mind (53). We will have

$$k_{20}^{(f)} = \frac{1}{2} Bq_{20}^{*(f)}\Big\{3[\sin^{-1} Bl - \sin^{-1} Bs + Bl\sqrt{1-B^2l^2} - Bs\sqrt{1-B^2s^2}] +$$
$$+ 2[Bl(1-B^2l^2)^{3/2} - Bs(1-B^2s^2)^{3/2}] +$$
$$+ \frac{8}{5Bl}[(1-B^2l^2)^{5/2} - (1-B^2s^2)^{5/2}]\Big\} \times$$
$$\times \Big\{\frac{7}{15}(1-B^2l^2)^{5/2} + \frac{1}{12}(1-B^2l^2)^{3/2} +$$
$$+ \frac{1}{8}\left(\sqrt{1-B^2l^2} + \frac{\sin^{-1} Bl}{Bl}\right) - \frac{4}{5}\Big\}^{-1}, \tag{57a}$$

$$k_{21}^{(f)} = \frac{1}{4} B^3lq_{20}^{*(f)}\Big\{3[\sin^{-1} Bl - \sin^{-1} Bs + Bl\sqrt{1-B^2l^2} - Bs\sqrt{1-B^2s^2}] +$$
$$+ 2[Bl(1-B^2l^2)^{3/2} - Bs(1-B^2s^2)^{3/2}] +$$
$$+ \frac{8}{5Bl}[(1-B^2l^2)^{5/2} - (1-B^2s^2)^{5/2}]\Big\} \times$$
$$\times \Big\{\frac{1}{4}B^2l^2\left[2(1-B^2l^2)^{3/2} + 3\left(\sqrt{1-B^2l^2} + \frac{\sin^{-1} Bl}{Bl}\right)\right] +$$
$$+ \frac{1}{5}((1-B^2l^2)^{5/2} - 1)\Big\}^{-1}. \tag{57b}$$

THE CASE OF CONCENTRATED SOURCES

Strating from equations (27a) and (27b), we will find, in in the case of concentrated sources, for Q_{20} and Q_{21} the following expressions:

$$Q_{20} = \frac{c}{(1-B^2c^2)^{3/2}}[(2-B^2c^2)(w_{10}^{(1)} - w_{10}^{(0)}) + c(w_{01}^{(1)} - w_{01}^{(0)})], \tag{58a}$$

$$Q_{21} = -\frac{1}{(1-B^2c^2)^{3/2}}[w_{10}^{(1)} - w_{10}^{(0)} + B^2c^2(w_{01}^{(1)} - w_{01}^{(0)})], \tag{58b}$$

and from (38a) and (38b), in which we will introduce U'_{11} given by (24), we will obtain

$$A_{20} + 2l^2 A_{22} + \frac{2}{\pi}\frac{l^2}{c^2}\sqrt{l^2 - c^2}\,(Q_{20} + 2cQ_{21}) = \frac{2}{\pi} w_{01}^{(0)} l^2, \tag{59a}$$

$$\begin{aligned}(A_{20} + A_{22}\, l^2)\frac{B^2}{1 - B^2 l^2}(E(k) - K(k)) + A_{22}\, E(k) - \\ - \frac{2}{\pi}\frac{\sqrt{l^2 - c^2}}{c^2}[Q_{20}\, B^2 c^2\, Y + cQ_{21}\,(Y - K(k)] = w_{10}^{(1)},\end{aligned} \tag{59b}$$

where Y is given by the expression (40a) for $s' = c$.

Using relation (33), we obtain

$$\frac{c}{l}\left(w_{10}^{(0)} + \frac{1}{2} c\, w_{01}^{(0)}\right) + \left(1 - \frac{c}{l}\right)\left(w_{10}^{(1)} + \frac{1}{2}(l + c)\, w_{01}^{(1)}\right) = w_{10} + \frac{1}{2} w_{01} l, \tag{60}$$

which, together with equation (58a), (58b), (59a), (59b) and (42), forms the system of equations from which we deduce the constants.

In this way, we will obtain

$$\frac{w_{10}^{(0)}}{w_{10}} = \frac{2\,B^2 c^2 \sqrt{l^2 - c^2}\,[(1 - B^2 c^2)\,\Pi\,(\rho, k) + K(k) - E(k)]}{2\,B^2 c^2 \sqrt{l^2 - c^2}\,[(1 - B^2 c^2)\,\Pi\,(\rho, k) + K(k) - E(k)] - \pi c\left(1 - \frac{c}{l}\right)(1 - B^2 c^2)^{3/2}}, \tag{61a}$$

$$\frac{w_{10}^{(1)}}{w_{10}} = 1 - \frac{\pi c\,\frac{c}{l}\,(1 - B^2 c^2)^{3/2}}{2\,B^2 c^2 \sqrt{l^2 - c^2}\,[(1 - B^2 c^2)\,\Pi\,(\rho, k) + K(k) - E(k)] - \pi c\left(1 - \frac{c}{l}\right)(1 - B^2 c^2)^{3/2}}, \tag{61b}$$

$$A_{20}^{(n)} = \frac{2\,B^2\,l^2\,c\sqrt{l^2 - c^2}}{\pi\,(1 - B^2\,c^2)^{3/2}}\,(w_{10}^{(0)} - w_{10}^{(1)}), \tag{61c}$$

$$A_{22}^{(n)} = -\,\frac{2\,B^2\,c\sqrt{l^2 - c^2}}{\pi\,(1 - B^2\,c^2)^{3/2}}\,(w_{10}^{(0)} - w_{10}^{(1)}), \tag{61d}$$

$$Q_{20}^{(n)} = -\,\frac{\pi\,c^2\,(2 - B^2\,c^2)\,w_{10}}{2B^2c^2\sqrt{l^2-c^2}\,[(1-B^2c^2)\,\Pi\,(\rho,k) + K\,(k) - E\,(k)] - \pi c\left(1-\dfrac{c}{l}\right)(1-B^2c^2)^{3/2}}, \tag{61e}$$

$$Q_{21}^{(n)} = \frac{\pi\,c\,w_{10}}{2B^2c^2\sqrt{l^2-c^2}\,[(1-B^2c^2)\,\Pi\,(\rho,k) + K\,(k) - E(k)] - \pi c\left(1-\dfrac{c}{l}\right)(1-B^2c^2)^{3/2}} \tag{61f}$$

in the case of the curved delta wing, and

$$\frac{w_{01}^{(0)}}{w_{01}} = \frac{(1 - 2\,B^2\,c^2)\,E\,(k) + B^2\,c^2\,K\,(k)}{(1 - B^2\,c^2)^{3/2}\sqrt{1 - \dfrac{c^2}{l^2}} + (1 - 2\,B^2\,c^2)\,E\,(k) + B^2\,c^2\,K(k)}, \tag{62 a}$$

$$\frac{w_{01}^{(1)}}{w_{01}} = \frac{(1 - B^2\,c^2)^{3/2} + \sqrt{1 - \dfrac{c^2}{l^2}}\,[(1 - 2B^2c^2)\,E\,(k) + B^2c^2\,K(k)]}{\left(1 - \dfrac{c^2}{l^2}\right)(1 - B^2\,c^2)^{3/2} + \sqrt{1 - \dfrac{c^2}{l^2}}\,[(1 - 2B^2c^2)\,E(k) + B^2c^2\,K\,(k)]}, \tag{62b}$$

$$A_{20}^{(f)} = -\,\frac{2}{\pi}\,l^2\left[w_{01}^{(0)}\,l - \frac{1 - 2\,B^2c^2}{(1 - B^2\,c^2)^{3/2}}\sqrt{l^2 - c^2}\,(w_{01}^{(1)} - w_{01}^{(0)})\right], \tag{62c}$$

$$A_{22}^{(f)} = \frac{2}{\pi}\left[w_{01}^{(0)}\,l - \frac{1 - 2\,B^2\,c^2}{(1 - B^2\,c^2)^{3/2}}\sqrt{l^2 - c^2}\,(w_{01}^{(1)} - w_{01}^{(0)})\right], \tag{62d}$$

$$Q_{20}^{(f)} = w_{01}\,\frac{c^2}{\left(1 - \dfrac{c^2}{l^2}\right)(1-B^2c^2)^{3/2} + \sqrt{1 - \dfrac{c^2}{l^2}}\,[(1-2B^2c^2)\,E\,(k) + B^2c^2K(k)]}, \tag{62e}$$

$$Q_{21}^{(f)} = -w_{01}\,\frac{B^2c^2}{\left(1 - \dfrac{c^2}{l^2}\right)(1-B^2c^2)^{3/2} + \sqrt{1 - \dfrac{c^2}{l^2}}\,[(1 - 2B^2c^2)\,E(k) + B^2c^2K(k)]} \tag{62f}$$

for the torsioned delta wing.

The constants k_{20}, k_{21} is deduced immediately from (45a), (45b) (47a), (47b), taking into consideration (53),(61b), (62a) (62b).

4. THE CALCULATION OF THE DISTRIBUTION OF PRESSURE AND AERODYNAMIC CHARACTERISTICS

We have shown above that the axis of disturbance velocity on the real wing results through the superpositioning of the three imaginary wing components, obtaining formula (23).

For the calculation of the distribution of pressure, the total axis velocity given by (3) (fig. 3) will be considered.

$$C_p = -2\frac{u_1}{U_\infty} = -2\,\mathrm{Re}\,\frac{\mathcal{U}_1}{U_\infty}. \qquad (63)$$

Moving along to the calculation of the coefficient of lift of the wing we will make the observation that the wings of symmetrical thickness do not give lift so that only the coefficient of lift of the "thin lift wing" will be taken into consideration:

$$C_z = \frac{8}{3lU_\infty}\int_0^l u_{11}\,dy. \qquad (64)$$

Keeping in mind(16), we will obtain the following expression of the coefficient of lift of the wing:

$$C_s = \frac{4l}{3U_\infty}\left\{\frac{\pi A_{20}}{2l^2} + q^\bullet_{20}\left[\cos^{-1}\frac{s}{l} - \frac{s}{l}\left(1-\frac{s^2}{l^2}\right)^{1/2} - \frac{2}{3}\left(1-\frac{s^2}{l^2}\right)^{3/2}\right] + \right.$$

$$\left. + \frac{1}{8}q^\bullet_{21}\, l\left[\left(\frac{8}{3}\left(1-\frac{s^2}{l^2}\right) - \frac{s}{l}\left(1-2\frac{s^2}{l^2}\right)\right)\left(1-\frac{s^2}{l^2}\right)^{1/2} - \cos^{-1}\frac{s}{l}\right]\right\}, \quad (6$$

in the case of distributed sources, and

$$C_s = \frac{8}{3U_\infty}\left[\frac{\pi A_{20}}{4l} + \left(Q_{20} + \frac{c}{2}Q_{21}\right)\sqrt{1-\frac{c^2}{l^2}}\right], \qquad (66)$$

in the hypothesis of concentrated sources in the point y = c on the wing.

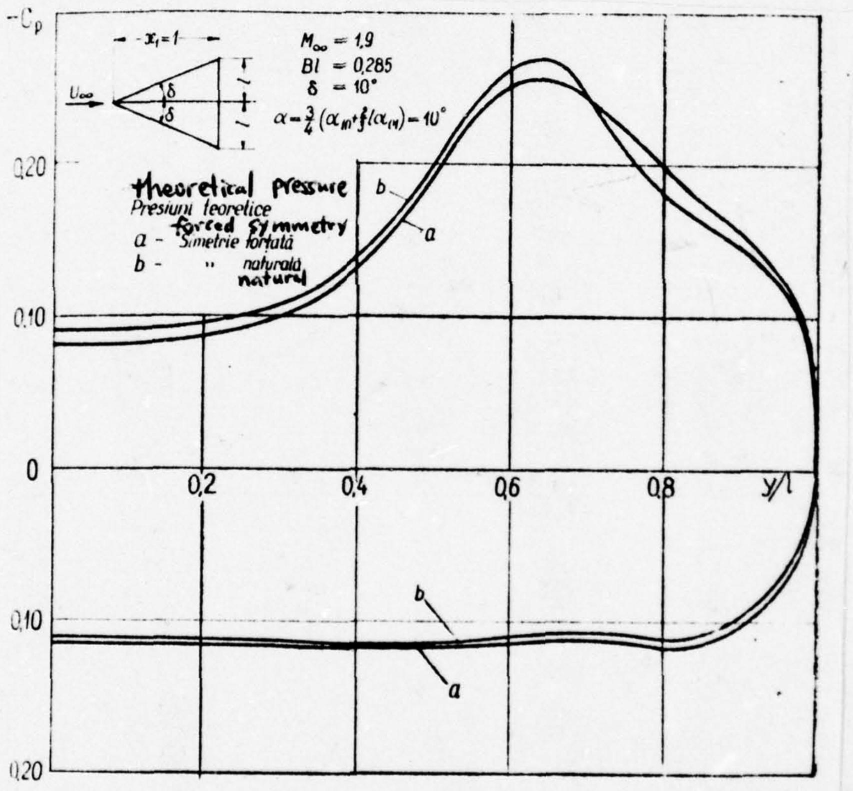

Fig. 3

In order to define the parameter $\frac{s}{l}$, which enters into the expressions above and which determines the limits of the distribution of sources, we will remark first that the position

of the maximum distribution of pressures coincides with the abscissa y = c of the center of the vortex nucleus, as is ascertained in other ways from experience. But, making the calculation on the basis of the distribution of sources (14), it is ascertained that the peak of the depressions on the higher side of the wing falls approximately at half the distance between the center of gravity of the intensities of the sources and the abscissa point s, on which we keep in view to determine them. In this way we can deduce the relation between s and c:

$$c = s + \frac{1}{6}(l - s) \qquad \left(\frac{s}{l} = 1{,}2\,\frac{c}{l} - 0{,}2\right). \tag{67}$$

Next, for the determination of the positions of the vortex nuclei centers, we will observe that, at curves and small torsions of the wings, these can be considered perfectly plane and parallel with the undisturbed flow U_∞, so that the vortexes falling off at the leading edges, in the form of horns, can be considered as absent from the higher surface of the wing. This work would set out, however small the intensity of the vortexes would be, at very great local velocities, incompatible with the real effects of the separation of flow at the edges. In order to avoid this matter, we must allow that with curved and very small torsion of the wings, accordingly for very small values of the parameter $(\alpha_{10} \pm l\alpha_{01})$ (3) or more exactly, when $\alpha_{10} \pm l\alpha_{01} \rightarrow 0$, the position of the vortex nucleus will be $c = l$.

For greater curves (torsion) on the wings we will allow that the nature of vortexes, which start on the leading edges, evolve proportionally with the incidence corresponding to respective sections, as well as with the span of the wing (as with the wing of constant incidence (1)), accordingly with the square of the span of the wing, and winds itself on an axis representing the line of the centers of gravity, which correspond at an abscissa

$$\frac{c}{l} = \frac{3}{4}. \qquad (68$$

We will remark however that the intensity of the vortex nucleus is something smaller than the total vortex generating intensity, the rest being in the thin layer which forms the surface of the horn along the leading edge.

From this, directing ourselves according to experiments accomplished on the delta wing with constant incidence, the minimum position of the vortex center, corresponding to curves (torsions) or large parameters $\alpha_{10} \pm l\alpha_{01}$, can be considered

$$c \cong 0{,}55\, l \div 0{,}65\, l. \qquad (69$$

What makes this minimum position ~~very stable~~ unstable is the fact that at great incidences interior vortex nuclei appear, of opposed direction with the principal, situated between this and the leading edge.

Between these limits, keeping in mind experimental results obtained by diverse authors, on the plane delta wing with constant incidence, we will allow for the position of the vortex nucleus the following approximate formula of variation with the incidence:

$$\frac{c}{l} \cong \frac{1}{1 + 1{,}4\,(\alpha)^{1/2}}, \qquad (70)$$

where α represents the incidence of the curved wing in a suitable section corresponding to the center of gravity of the aerodynamic effects, namely

$$x_1 = \frac{3}{4}. \qquad (71)$$

For the torsioned wing (with forced symmetry), due to the variation of incidences in section x_1 in function x_2, we will consider like the incidence calculated in the formula (70), the point

$$x_1 = \frac{3}{4}, \quad x_2 = \frac{2}{3}\,l\,x_1 = \frac{1}{2}\,l. \qquad (72)$$

We will remark that the whole rational which led to stability formula (70) can be applied to each section of the wing, obtaining in this way a curved line for the positions of the vortex nuclei. However, deviations from a straight line are ascertained at small incidences, towards the peak of the wing, where the contribution of lift power of the wing and the distribution of pressures is small due to small surface and incidence, and in the back of the

wing, where the incidences grow, the vortexes are situated rectiliniar. In this ~~was~~ way we have succeeded in planning systematically a vortex sheet in the form of a horn, on which an axis - being considered a straight line - of which the effects on the wing ~~of~~ by the vortex nuclei are found, introduced through distributed sources. This work was necessary because only in this manner can we apply the methods of conical flow of the higher order (2). In opposed cases, the nuclei being situated on a curved line, a quasiconical vortex sheet results, and therefore motion becomes, as such, more complicated.

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THE THIN DELTA WING, WITH VARIABLE GEOMETRY, OPTIMUM FOR TWO SUPERSONIC CRUISING SPEEDS*)

by ADRIANA NASTASE

Institute of Fluid Mechanics

Academy of the Socialist Republic of Romania

In the present article, the form of thin component surfaces of the heavy lift delta wing with variable geometry, is determined, which has the wing span $2\,l$ given when it is completely folded (fig. 1) and which must be optimum when it is placed in the flow characterized through the MACH number M, and when it is completely unfolded (fig. 2) it has the wing span $2\,L > 2\,l$ and must be optimum when it is placed in the flow characterized through the Mach number $M_1 < M$.

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*) The following principals presented in this paper were communicated at the Military Technical Academy in Brno. Nov. 1968, under the title "The Thin Delta Wing, with Variable Geometry, Optimum for two Supersonic Cruising Speeds".

The designs for the thin delta wing with variable geometry, the lift powers and moments of dive in the folded and unfolded position, as well as the form of the central airfoil are given. In order to avoid the falling off of flow on the subsonic leading edges we will cancel the axis of disturbance velocity u along the subsonic leading edges of the wing in the folded and unfolded position. Similarly, the wing in unfolded position has to be continued at the traverse of the edges of the folded wing (becoming edges on the unfolded wing). Through the application of variational methods, the problem of the form of the surface of the thin delta wing optimum for two supersonic cruising speeds is reduced to the resolving of an algebraic system of liniar equations.

1. GENERAL CONSIDERATIONS. THE REDUCTION OF THE PROBLEM OF EXTREMUM OF THE THIN DELTA WING WITH VARIABLE GEOMETRY, PROPOSED BY THE STUDY OF TWO VARIATIONAL PROBLEMS IN CASCADE REFERING TO THE FOLDED AND UNFOLDED WING

From investigations of specialized liturature we have never found another theoretical paper refering to the determination of the form of the surface of wings with variable geometry, optimum for two supersonic cruising speeds. On the other hand, wings with variable geometry present a special interest in modern

aviation, since they have a series of advantages over the wings of the invariable form, advantages among which we mention the most important:

-- facilitates the landing reducing the length of the runway and the landing velocity;

-- permits the increase of lift capacity of the wing at great velocities;

-- permits the control of aceleration and thermic control exit and entrance in the dense atmosphere.

We will study in this article the form of the surface of thin delta wings, with variable geometry, optimum for two supersonic cruising speeds. This type of wing can be found being utilized especially in aerospace vessels which exit and reenter in the dense atmosphere.

In the present work we begin the study of the heavy lift delta wing with variable geometry, optimum for two supersonic cruising speeds, eventually provided with an edge of central separation for which are given: the layout, lift powers, the moments of dive and volumes of the wing in the folded and unfolded position, the form of the central airfoil, as well as a series of geometric conditions enclosing the wing along its leading and back edges, continuity of height of the unfolded wing along the

leading edges of the folded wing (becoming edges on the folded wing). Similarly, the axes of disturbance velocities must be nought along the leading edges of the folded and unfolded wings in order to avoid at the two cruising speeds the birth of horn shaped vortexes, which have the tendency to form along the subsonic leading edges.

Since as in the framework of liniar theory the effect of lift power can be separated from the effect of weight in the folded position as well as in the unfolded position, we will have therefore to study the thin component and that of symmetrical thickness separately.

The study of the thin delta wing with variable geometry is reduced to resolving the following two problems of extremum in cascade: first, in which the optimum form of the surface of the folded wing (of wingspan 2 l) (fig. 1a,b) is determined at the cruising speed characterized through the Mach number M, and the second, in which the optimum form of the added surface S (which is presented, in our case, under the form of whirls at the leading edge) is determined, in such a manner, that the whole unfolded delta wing (of wingspan 2 L) (fig. 2a,b,c) will be optimum at the cruising speed characterized through the Mach number M_1.

The first variational problem consists therefore in the determiniation of the form of the surface of a thin delta wing which has the resistance at minimum advance at the flight speed characterized through the Mach number M and for which are given the projection in the plane of the wing (represented through isoscles triangle OA_1A_2), the lift power C_l, the moment of dive C_m and the central airfoil. In addition, we will consider that the axis of disturbance velocity u is canceled along the subsonic leading edges OA_1 and OA_2 in order to prevent, at the considered cruising speed, the birth of vortexes in the form of horns. In the calculation of the resistance at the advance we will also include the forces of suction ~~sections~~ which appear on the subsonic leading edges of folded thin wings.

Also, we will presuppose, for generality, that the wing is provided with a central edge OC.

The second variational problem consists in determining the form of the added surface S (shown in fig. 2a,b,c), such that the whole unfolded wing will have the minimum resistance at the advance, at the cruising speed characterized through the Mach number $M_1<M$ and for which are given: the design of the wing (represented by isosclees triangle $OA'_1A'_2$), lift power C_L and the moment of dive C_M. In addition, we will impose the condition that, along line OA_1 and OA_2 (the edges of the folded wing which

become edges on the unfolded wing), the height of the surface of the wing will be continuous, and the axis of disturbance velocity u will be canceled along the subsonic leading edges OA_1' and OA_2'. Similarly, we will include the advance resistance and the forces of suction that appear on the subsonic leading edges OA_1' and OA_2' of the unfolded thin wing.

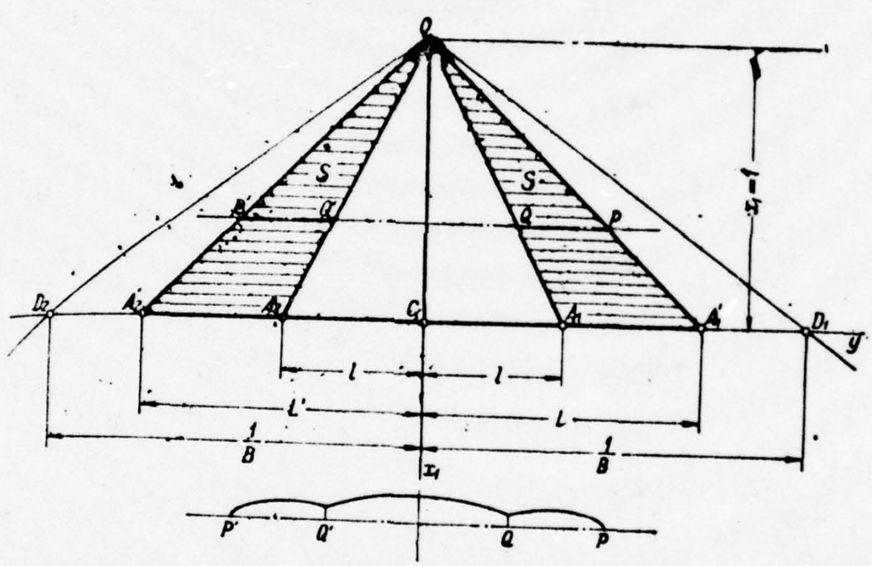
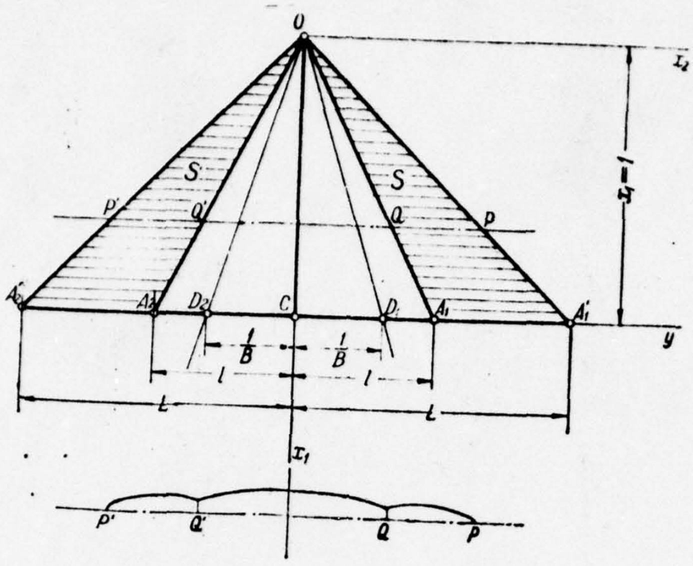
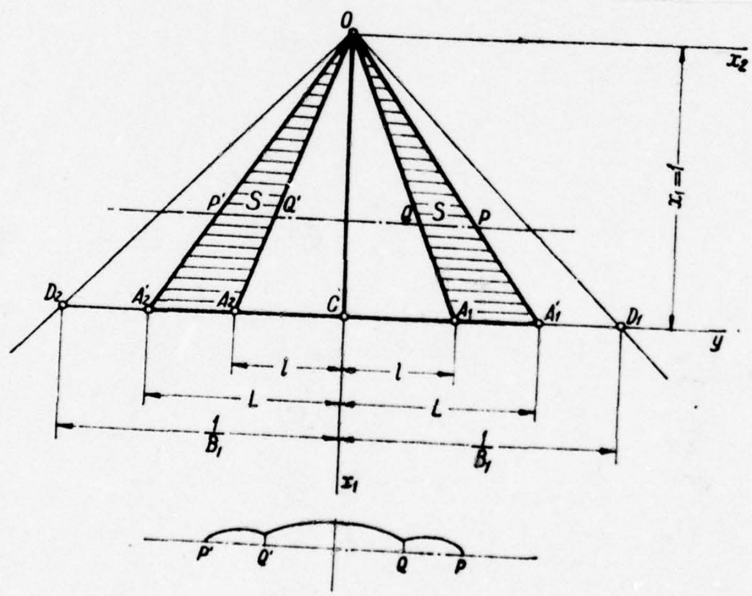


Fig. 1a



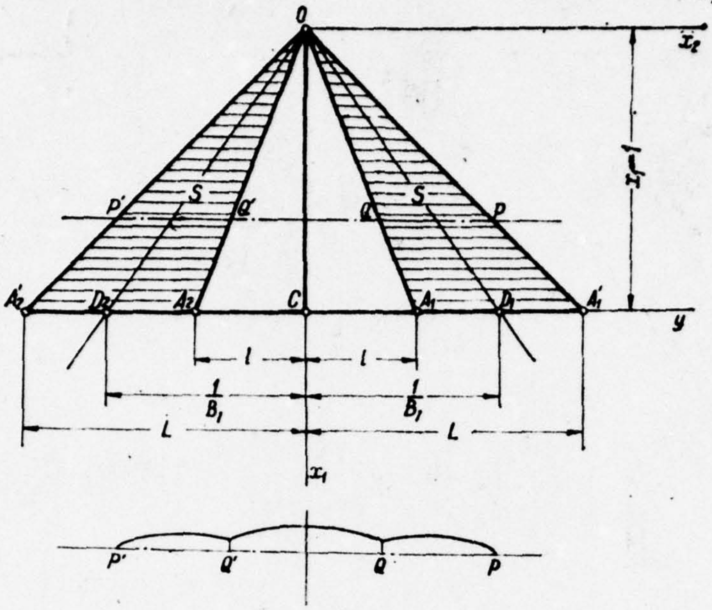
O
x_2
$x_1=1$
S
S
P'
Q'
Q
P
A_2'
A_2
D_2
C
D_1
A_1
A_1'
y
$\frac{1}{B}$
$\frac{1}{B}$
l
l
L
L
x_1
P'
Q'
Q
P

Fig. 1b



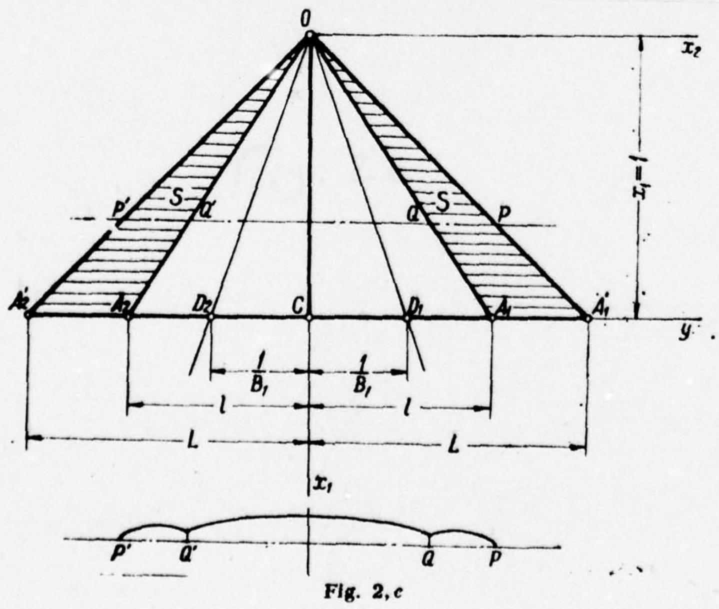
O
x_2
$x_1=1$
P'
S
Q'
Q
S
P
D_2
A_2'
A_2
C
A_1
A_1'
D_1
y
l
l
L
L
$\frac{1}{B_1}$
$\frac{1}{B_1}$
x_1
P'
Q'
Q
P

Fig. 2a



0
x_2
$x_1 = 1$
P'
S
Q'
Q
S
P
A'_2
D_2
A_2
C
A_1
D_1
A'_1
y
l
l
$\frac{1}{B_1}$
$\frac{1}{B_1}$
L
L
x_1
P'
Q'
Q
P

Fig. 2b



0
x_2
$x_1 = 1$
P'
S
Q'
Q
S
P
A'_2
A_2
D_2
C
D_1
A_1
A'_1
y
$\frac{1}{B_1}$
$\frac{1}{B_1}$
l
l
L
L
x_1
P'
Q'
Q
P

Fig. 2, c

As in previous works (1) - (6), refering to the determination of optimum forms of the surfaces of thin lift power systems, we will treat the two above mentioned variational problems in the following two hyposthesees of calculation:

-- the first hypothesis, somewhat classic, in which the axis of disturbance velocity u is infinite on the subsonic leading edges, suction appears and the effect of the falling off at the leading edges is omitted; this hypothesis is valid at small incidences;

-- the second hypothesis, which, at the chosen flight speed, we avoid the falling off of flow on the subsonic leading edges, through the cancelation of the axis of disturbance velocity u along that edge. In this case, the effect of conturing of the subsonic leading edges of the thin wing disappears and once again, with the disappearance of this effect, the forces of suction also disappear which grew precisely due to that effect.

As has been shown above in the introduction of the two variational problems, the advance resistances and the suction are also included among the conditions of relation of the problems, and the conditions of cancelation of the axis of disturbance velocity u along the subsonic leading edges.

As a result, if inthe final systems, which give the solutions

of the two problems of extremum mentioned above, we exclude the conditions of cancelation of the axes of disturbance velocity on the subsonic leading edges, we place ourselves in the position of the first hypothesis of calculation. If however we exclude from the final systems of terms resulting from suction, we will place ourselves in the second hypothesis of calculation.

2. THE EXPRESSIONS OF THE AXIS OF DISTURBANCE VELOCITY u ON THE FOLDED AND UNFOLDED WING

We will consider the wing referred to a triorthogonal system of axes $Ox_1x_2x_3$, which has axis Ox_1 parallel with the direction of flow from infinity U_∞ and the wing is considered plotted in the plane Ox_1x_2. We will presuppose that the vertical velocities of disturbance w and $\bar{w}$ on the wing are expressed in the form of superimposing homogeneous polinomials w_{m-1} and $\bar{w}_{m-1}$ in x_1 and x_2, that is:

$$w = \sum_{m=1}^{N} w_{m-1} = \sum_{m=1}^{N} x_1^{m-1} \sum_{k=0}^{m-1} w_{m-k-1.k} |y|^k \qquad \left(y = \frac{x_2}{x_1}\right) \qquad (1)$$

on OA_1A_2 and

$$\bar{w} = \sum_{m=1}^{N} \bar{w}_{m-1} = \sum_{m=1}^{N} x_1^{m-1} \sum_{k=0}^{m-1} \bar{w}_{m-k-1.k} |y|^k \qquad (2)$$

on OA_2A_2' and OA_1A_1'. In this formula, $w_{m-k-1.k}$ and $\bar{w}_{m-k-1.k}$ are the constants and $N - 1$ isthe rank of the polinomial of highest order taken into consideration in the expressions w and $\bar{w}$.

If we integrate expressions (1) and (2) in reference with x_1, we obtain heights $Z(x_1,x_2)$ and $\overline{Z}(x_1,x_2)$ on OA_1A_2 and OA_2A_2' respectively and OA_1A_1' :

$$Z(x_1, x_2) = \frac{1}{U_\infty}\int w \, dx_1 + f(x_2) =$$

$$= \frac{1}{U_\infty}\sum_{m=1}^{N} x_1^m \sum_{k=0}^{m-1} w_{m-k-1,k} \frac{|y|^k}{m-k} + f(x_2) \quad \left(y = \frac{x_2}{x_1}\right) \tag{3}$$

and

$$\overline{Z}(x_1, x_2) = \frac{1}{U_\infty}\int \overline{w} \, dx_1 + \overline{f}(x_2) =$$

$$= \frac{1}{U_\infty}\sum_{m=1}^{N} x_1^m \sum_{k=0}^{m-1} \overline{w}_{m-k-1,k} \frac{|y|^k}{m-k} + \overline{f}(x_2). \tag{4}$$

In the above formulae, $f(x_2)$ and $\overline{f}(x_2)$ are some arbitrary functions of x_2 which we can eventually use in the placing of supplementary geometric conditions on the surface of the wing.

Considering first the folded wing (closely) (fig. 1) having OA_1 and OA_2 as leading edges and OC as central edges, the expression of the axis of disturbance velocity u on the folded wing will be (7)

$$u = \sum_{n=1}^{N} x_1^{n-1} \left\{ \frac{\sum_{q=0}^{E\left(\frac{n}{2}\right)} A_{n,2q}\, y^{2q}}{\sqrt{B^2(l^2 - y^2)}} + \sum_{q=1}^{E\left(\frac{n-1}{2}\right)} C_{n,2q}\, y^{2q} \operatorname{argch} \sqrt{\frac{l^2}{y^2}} \right\} \tag{5}$$

if the folded wing has subsonic leading edges and

$$u = \sum_{n=1}^{N} x_1^{n-1} \left\{ \frac{\sum_{q=0}^{E\left(\frac{n}{2}\right)} A_{n,2q}\, y^{2q}}{\sqrt{1 - B^2 y^2}} + \sum_{q=1}^{E\left(\frac{n-1}{2}\right)} C_{n,2q}\, y^{2q} \operatorname{argch} \sqrt{\frac{1}{B^2 y^2}} + \right.$$

$$+ \sum_{q=0}^{n-1} G_{nq}\, y^q \left[\arccos \sqrt{\frac{(1 + Bl)(1 - By)}{2B(l - y)}} + \right.$$

$$\left.\left. + (-1)^q \arccos \sqrt{\frac{(1 + Bl)(1 + By)}{2B(l + y)}} \right] \right\} \qquad (6)$$

for the folded wing with both supersonic leading edges.

We will now consider the unfolded wing (fig. 2). Since the geometry of the central portion of the wing is invariable, it results that $w = \frac{\partial z}{\partial x_1}$ will be invariable on portion OA_1A_2. The former leading edges OA_1 and OA_2 become edges on the new unfolded wing, and the new leading edges are now OA_1' and OA_2'. With these observations, the velocity of disturbance u on the unfolded wing will be of the form

$$u = \sum_{n=1}^{N} x_1^{n-1} \left\{ \frac{\sum_{q=0}^{E\left(\frac{n}{2}\right)} A'_{n,2q}\, y^{2q}}{\sqrt{B_1^2 (L^2 - y^2)}} + \sum_{q=1}^{E\left(\frac{n-1}{2}\right)} C'_{n,2q}\, y^{2q} \operatorname{argch} \sqrt{\frac{L^2}{y^2}} + \right.$$

$$+ \sum_{q=0}^{n-1} G'_{n,q}\, y^q \left[\operatorname{argch} \sqrt{\frac{(L + l)(L - y))}{2L(l - y)}} + \right.$$

$$\left.\left. + (-1)^q \operatorname{argch} \sqrt{\frac{(L + l)(L + y)}{2L(l + y)}} \right] \right\} \qquad (7)$$

if the edges OA_1' and OA_2' are subsonic or

$$u = \sum_{n=1}^{N} x_1^{n-1} \left\{ \frac{\sum\limits_{q=0}^{E\left(\frac{n}{2}\right)} A'_{n,2q}\, y^{2q}}{\sqrt{1 - B_1^2 y^2}} + \sum_{q=1}^{E\left(\frac{n-1}{2}\right)} C'_{n,2q}\, y^{2q} \operatorname{argch} \sqrt{\frac{1}{B_1^2 y^2}} + \right.$$
$$+ \sum_{q=0}^{n-1} G'_{nq}\, y^q \left[\operatorname{argch} \sqrt{\frac{(1 + B_1 l)(1 - B_1 y)}{2B_1(l - y)}} + \right.$$
$$\left. + (-1)^q \operatorname{argch} \sqrt{\frac{(1 + B_1 l)(1 + B_1 y)}{2B_1(l + y)}} \right] +$$
$$+ \sum_{q=0}^{n-1} H'_{nq}\, y^q \left[\arccos \sqrt{\frac{(1 + B_1 L)(1 - B_1 y)}{2B_1(L - y)}} + \right.$$
$$\left. \left. + (-1)^q \arccos \sqrt{\frac{(1 + B_1 L)(1 + B_1 y)}{2B_1(L + y)}} \right] \right\} \qquad (8)$$

if the leading edges OA_1' and OA_2' are supersonic.

Observations:

a) In formulae (6) and (8) the first term has enlarged itself with $\sqrt{1-B^2y^2}$ and $\sqrt{1-B_1^2y^2}$ respectively in order for the radical to appear in the denominator and so that we can be able to better systematize the calculation of the aerdynamic characteristics of the wing.

b) If the edges of the folded wing, as well as those of the unfolded wing are supersonic for the Mach number of flight M_1, then in terms which contain the coefficient G'_{nq} of formula (8) and which represent the contribution of the edges of the folded wing in the expression u "argch" is replaced with "argcos".

Coefficients $A_{n.2q}$, $C_{n.2q}$ etc. of the axes of disturbance velocities u are related to coefficients of the vertical velocities of disturbance through the liniar relations deduced from the compatibility conditions of P. Germain (8). Coefficients of the axes of disturbance velocities u on the folded wing will depend only on ceefficients $w_{n-j-1.j}$ of w, since they are functions only of the form of the surface of the folded wing. Therefore we can write

$$A_{n.2q} = \sum_{j=0}^{n-1} a_{2q.j}^{(n)} w_{n-j-1.j}, \tag{9a}$$

$$C_{n.2q} = \sum_{j=0}^{n-1} c_{2q.j}^{(n)} w_{n-j-1.j}, \tag{9b}$$

$$G_{nq} = \sum_{j=0}^{n-1} g_{qj}^{(n)} w_{n-j-1.j}. \tag{9c}$$

Instead, coefficients of the axis velocity of the unfolded wing will depend on coefficients of w, as well as, $\bar{w}$. Therefore we will have

$$A'_{n.2q} = \sum_{j=0}^{n-1} (a_{2q.j}^{\prime(n)} w_{n-j-1.j} + \bar{a}_{2q.j}^{\prime(n)} \bar{w}_{n-j-1.j}), \tag{10a}$$

$$C'_{n.2q} = \sum_{j=0}^{n-1} (c_{2q.j}^{\prime(n)} w_{n-j-1.j} + \bar{c}_{2q.j}^{\prime(n)} \bar{w}_{n-j-1.j}), \tag{10b}$$

$$G'_{n.q} = \sum_{j=0}^{n-1} (g_{qj}^{\prime(n)} w_{n-j-1.j} + \bar{g}_{qj}^{\prime(n)} \bar{w}_{n-j-1.j}), \tag{10c}$$

$$H'_{nq} = \sum_{j=0}^{n-1} (h_{qj}^{\prime(n)} w_{n-j-1.j} + \bar{h}_{qj}^{\prime(n)} \bar{w}_{n-j-1.j}). \tag{10d}$$

Observations.

a) in this last expression, $h'^{(n)}_{qj} = 0$ since the constants H'_{nq} obtained calculating a semiresidue around point $y = L$, thus only depend on the drop of vertical velocity of disturbance from this point, which is $-\bar{w}$. For homogenity we have written therefore the expressions of the coefficients H'_{nq} under the form (10d).

b) In formulae (9a), (9b) and (9c), coefficients $a^{(n)}_{2q.j}$, $c^{(n)}_{2q.j}$ and $g^{(n)}_{qj}$ depend only on the form in the plane of the folded wing (thus of l) and on the chosen flight velocity characterized by the Mach number M at which the wing is folded $(B=\sqrt{M^2-1})$.

c) Coefficients $a'^{(n)}_{2q.j}$, $\bar{a}'^{(n)}_{2q.j}$ etc. depend on the form in the plane of the folded wing, as well as the form in the plane of the unfolded wing, thus by l and L, as well as on the Mach number of flight M_1 at which the wing is in the unfolded position $(B=\sqrt{M_1^2-1})$. Therefore, for some designs given and for some flight velocities given for the folded wing, as well as for the unfolded wing, coefficients $a^{(n)}_{2q.j}$, $a'^{(n)}_{2q.j}$, $\bar{a}'^{(n)}_{2q.j}$ etc. are constants in the study of the two problems of extremum proposed. As a result, from the first variational problem, refering to the folded wing, we will determine the optimum values of the coefficients $w_{n-j-1.j}$ of w, and from the second variational problem, refering to the unfolded wing, we will determine the optimum values of the coefficients $\bar{w}_{n-j-1.j}$ of the vertical velocity of disturbance $\bar{w}$. In the second variational problem, coefficients $w_{n-j-1.j}$

of the vertical velocity of disturbance w, determined by the first variational problem, will be reckoned constant.

We will press on now to solve successively the two above mentioned variational problems. We will consider the geometric conditions of relation and aerodynamic characteristics only on half the wing, due to the symmetry of the wings.

3. THE STUDY OF THE FIRST VARIATIONAL PROBLEMS REFERING TO THE FOLDED WING

Before moving along to the solution of the first variational problems, we will explain the geometric conditions of relation and the aerodynamic characteristics that enter in the problem of extremum of the folded delta wing. In this direction we will use a series of results obtained before for the thin delta wing of minimum resistance provided with a central edge.

3.1 EXPLAINATION OF THE GEOMETRIC CONDITIONS OF RELATION OF THE FOLDED WING

The geometric conditions of relation for the folded wing are:

a) the condition that the central airfoil of the wing be imposed;

b) the condition that the axis of disturbance velocity be

canceled along the subsonic leading edges of the thin wing.

a) From the condition that the central airfoil be imposed ($Z(x_1,0)=\varphi(x_1)$) through the development inthe Mac Laurin series of $\varphi(x_1)$ and identification with $Z(x_1,0)$ we obtain the relations

$$f(0) = \varphi(0), \tag{11a}$$

$$R_N(x_1) = 0, \tag{11b}$$

$$E_t \equiv w_{t-1,0} - \frac{U_\infty}{(t-1)!}\varphi^{(t)}(0) = 0, \qquad t = 1, 2, \ldots N. \tag{11c}$$

The first relation (11a) gives us the value of f(0), the second (11b) indicates that the central airfoil has to be given in the form of a polynomial of rank N, and the relations (11c) enter as relations of connection in the studied problem of extremum.

From the condition of canceling the axis of disturbance velocity u along the subsonic leading edge OA_1 ($y=\ell$, $u=0$) we obtain, similar with (6), the relations

$$F_1^{(t)} \equiv \sum_{j=0}^{t-1} \Theta_{tj}\, w_{t-j-1,j} = 0, \tag{12a}$$

$$\Theta_{tj} = \sum_{q=0}^{E\left(\frac{t}{2}\right)} l^{2q} a_{2q,j}^{(t)}, \qquad t = 1, 2, \ldots, N. \tag{12b}$$

Observations.

To assure that the symmetry of the surface of the folded wing in reference with plane Ox_1x_3 and as a result continuity of the height of the surface at the traverse of the edge from

the origin, it is encessary here that the function $f(x_2)$, which enters into expression (3) of the height of the surface, be parried in reference with x_2, in other words to have

$$f(x_2) = f(-x_2). \tag{13}$$

3.2. EXPAINATION OF THE AERODYNAMIC CHARACTERISTICS OF THE FOLDED WING

The aerodynamic characteristics which enter in the problem of extremum of the folded wing are:

a) the lift power C_1;

b) the moment of dive C_m;

c) the wave resistance C_d;

d) the coefficient of suction σ_0.

a) The condition of the lift power of the wing to be given C_1 = const becomes

$$C_l = \sum_{n=1}^{N} \sum_{j=0}^{n-1} \Lambda_{nj} w_{n-j-1,j} = \text{const}. \tag{14}$$

In this formula Λ_{nj} are constants in the studied problem of extremum and which are of the form

$$\Lambda_{nj} = \frac{8}{lU_\infty(n+1)} \left\{ \sum_{q=0}^{E\left(\frac{n}{2}\right)} a_{2q,j}^{(n)} \mathcal{J}_{2q}(0, l) + \right.$$

$$\left. + Bl \sum_{q=1}^{E\left(\frac{n-1}{2}\right)} \frac{c_{2q,j}^{(n)}}{2q+1} \mathcal{J}_{2q}(0, l) \right\} \tag{15}$$

if the delta wing has subsonic leading edges and, respectively,

$$\Lambda_{nj}^{**}=\frac{8}{lU_\infty(n+1)}\left\{\sum_{q=0}^{E\left(\frac{n}{2}\right)} a_{2q,j}^{(n)}\mathcal{J}_{2q}^{**}\left(0,\frac{1}{B}\right)+\sum_{q=1}^{E\left(\frac{n-1}{2}\right)}\frac{c_{2q,j}^{(n)}}{2q+1}\mathcal{J}_{2q}^{**}\left(0,\frac{1}{B}\right)+\right.$$

$$+\sum_{q=0}^{n-1}\frac{g_{qj}^{(n)}}{q+1}\left[\frac{\sqrt{B^2l^2-1}}{2}\left(1+(-1)^q\right)\sum_{t=0}^{q} l^{q-t}\mathcal{J}_t^{**}\left(0,\frac{1}{B}\right)+\right.$$

$$\left.\left.+\,l^{q+1}\arccos\sqrt{\frac{1}{B^2l^2}}\right]\right\} \qquad (16)$$

if the delta wing has supersonic leading edges.

b) the condition of the mement of dive of the wing to be given becomes

$$C_m=\sum_{n=1}^{N}\sum_{j=0}^{n-1}\Gamma_{nj}w_{n-j-1,j}=\sum_{n=1}^{N}\sum_{j=0}^{n-1}\frac{n+1}{n+2}\Lambda_{nj}w_{n-j-1,j}=C_{m_0}. \qquad (17)$$

In this formula Λ_{nj} have meaning from (14).

c) The expression of the coefficient of wave resistance becomes

$$C_x=\sum_{n=1}^{N}\sum_{m=1}^{N}\sum_{j=0}^{n-1}\sum_{k=0}^{m-1}\Omega_{nmkj}w_{n-j-1,j}w_{m-k-1,k}, \qquad (18)$$

where Ω_{nmkj} are the following constants:

$$\Omega_{nmkj}=\frac{8}{U_\infty^2 l(m+n)}\left\{\sum_{q=0}^{E\left(\frac{n}{2}\right)} a_{2q,j}^{(n)}\mathcal{J}_{k+2q}(0,l)+\right.$$

$$\left.+\,Bl\sum_{q=1}^{E\left(\frac{n-1}{2}\right)}\frac{c_{2q,j}^{(n)}}{k+2q+1}\mathcal{J}_{k+2q}(0,l)\right\} \qquad (19$$

for the folded delta wing with subsonic leading edges and, respectively,

$$\Omega_{nmkj}^{**}=\frac{8}{U_\infty^2 l(m+n)}\left\{\sum_{q=0}^{E\left(\frac{n}{2}\right)} a_{2q,j}^{(n)}\mathcal{J}_{k+2q}^{**}\left(0,\frac{1}{B}\right)+\right.$$

$$+\sum_{q=1}^{E\left(\frac{n-1}{2}\right)} \frac{c_{2q,j}^{(n)}}{k+2q+1} \mathcal{J}_{k+2q}^{**}\left(0, \frac{1}{B}\right) + \sum_{q=0}^{n-1} \frac{g_{qj}^{(n)}}{k+q+1}\left[\frac{\sqrt{B^2l^2-1}}{2}\left(1 + \right.\right.$$

$$\left.\left. + (-1)^q\right) \sum_{t=0}^{k+q} l^{k+q-t} \mathcal{J}_t^{**}\left(0, \frac{1}{B}\right) + l^{k+q+1} \arccos \sqrt{\frac{1}{B^2l^2}}\right]\Bigg\} \qquad (2$$

for the folded delta wing with supersonic edges.

In formulae (15), (16), (19) and (20) the integrals $\mathcal{J}_k$ and $\mathcal{J}_k^{**}$ have the following values

$$\mathcal{J}_{2t}(0, l) = \int_0^l \frac{y^{2t}\,\mathrm{d}y}{\sqrt{B^2(l^2-y^2)}} = \frac{\pi(2t)!\,l^{2t}}{2^{2t+1}(t!)^2 B}, \qquad \text{pentru } k = 2t, \quad (21a)$$

$$\mathcal{J}_{2t+1}(0, l) = \int_0^l \frac{y^{2t+1}\,\mathrm{d}y}{\sqrt{B^2(l^2-y^2)}} = \frac{2^{2t}(t!)^2\, l^{2t+1}}{(2t+1)!\,B}, \qquad \text{pentru } k = 2t+1, \quad (21b)$$

$$\mathcal{J}_{2t}^{**}\left(0, \frac{1}{B}\right) = \int_0^{\frac{1}{B}} \frac{y^{2t}\,\mathrm{d}y}{\sqrt{1-B^2y^2}} = \frac{\pi(2t)!}{2^{2t+1}\,(t!)\;B^{2t+1}}, \qquad \text{pentru } k = 2t, \quad (22a)$$

$$\mathcal{J}_{2t+1}\left(0, \frac{1}{B}\right) = \int_0^{\frac{1}{B}} \frac{y^{2t+1}\,\mathrm{d}y}{\sqrt{1-B^2y^2}} = \frac{2^{2t}(t!)^2}{(2t+1)!\,B^{2t+2}}, \qquad \text{pentru } k = 2t+1 \quad (22b)$$

d) in conclusion, similar with (1), (9), the expression of the coefficient of suction will be of the form

$$\sigma = \sum_{n=1}^{N} \sum_{m=1}^{N} \sum_{k=0}^{m-1} \sum_{j=0}^{n-1} \Delta_{nmkj}\, w_{m-k-1,k}\, w_{n-j-1,j}, \qquad (23)$$

where

$$\Delta_{mmkj} = \frac{4\pi}{U_\infty^2\,(m+n)} \sum_{q=0}^{E\left(\frac{n}{2}\right)} \sum_{q'=0}^{E\left(\frac{m}{2}\right)} l^{2(q+q')} \sqrt{1-B^2l^2}\; a_{2q,j}^{(n)}\, a_{2q',k}^{(n)}. \qquad (24)$$

3.3 DETERMINATION OF THE OPTIMUM VALUE OFTHE VERTICAL VELOCITY OF DISTURBANCE w FOR THE FOLDED WING

We have accordingly found

$$C_d = \text{minim} \qquad (25)$$

with the condition of connection

$$C_l = C_{l_0}, \quad C_m = C_{m_0}, \quad E_t = 0, \quad F_1^{(t)} = 0, \qquad t = 1,2 \ldots N; \qquad (26)$$

similar with (1) - (6), the calculation of this extremum with connections is reduced to the calculation of the extremum without connections of the expression

$$H = (C_d + \sigma) + \lambda^{(1)} C_l + \lambda^{(2)} C_m + \sum_{t=1}^{N} (\lambda_t E_t + \mu_t F_1^{(t)}). \qquad (27)$$

Canceling, as in (1) - (6), coefficients of the variations $\delta w_{0\sigma}$, we obtain the following equations:

$$\sum_{n=1}^{N} \sum_{j=0}^{n-1} (\Omega_{n,0+\sigma+1,\sigma,j} + \Omega_{0+\sigma+1,n,j\sigma} + \Delta_{n,0+\sigma+1,\sigma,j} +$$
$$+ \Delta_{0+\sigma+1,n,j,\sigma}) w_{n-j-1,j} + \lambda^{(1)} \Lambda_{0+\sigma+1,\sigma} + \lambda^{(2)} \Gamma_{0+\sigma+1,\sigma} +$$
$$+ \lambda_{0+1} \delta_0^\sigma + \mu_{0+\sigma+1} \Theta_{0+\sigma+1,\sigma} = 0,$$
$$0, \sigma = 0,1 \ldots (N-1), \qquad 1 \leqslant 0 + \sigma + 1 \leqslant N. \qquad (28)$$

These equations, together with the conditions of connection (26), form a liniar algebraic system which determines in a certain way the values of coefficients $w_{\theta\sigma}$ of w, as well as the values of (Lagrangeen) the multiplicators λ_t and u_t.

The first variational problem refering to the folded wing can therefore be considered solved.

4. THE STUDY OF THE TWO VARIATIONAL PROBLEM REFERING TO THE UNFOLDED DELTA WING

As in the case of the first variational problem, we will explain first the geometric conditions of connection and the aerodynamic characteristics that enter in the formulation of the problem of extremum for the unfolded wing. After that we

will consider the central portion of the wing, that is, the folded delta wing A_1OA_2 given by the first variational problem, and we will determine the form of the added surface composed of two whirls by the leading edge (formed from triangles A_1OA_1' and A_2OA_2'), in such a way that the whole unfolded wing will be at minimum resistance at the second cruising speed characterized by the Mach number M_1.

4.1 THE EXPLAINATION OF THE GEOMETRIC CONDITIONS OF CONNECTION OF THE FOLDED WING

In the case of the unfolded delta wing, we will impose the following conditions of connection of natural geometry:

a) the condition of continuity of height of the surface of the wing at the traverse of the edges OA_1 and OA_2 (which are the leading edges of the folded wing);

b) the conditions of cancelation of the axis of disturbance velocity u along the subsonic leading edges OA_1' and OA_2' of the unfolded wing.

Because of the symmetry of the wing, the above mentioned conditions will be put only on edge OA_1 and on the edge OA_1'.

a) From the condition of continuity of height of the surface of the unfolded wing along edge OA_1 ($y=1, Z=\bar{Z}$) is obtained the relation

$$\frac{1}{U_\infty}\sum_{m+1}^{N} x_1^m \sum_{k=0}^{m-1} \frac{l^k}{m-k}(w_{m-k-1,k} - \overline{w}_{m-k-1,k}) = f(x_1 l) - \overline{f}(x_1 l), \quad (29$$

which is similar with that established earlier (10) (in which we had to substitute s = l).

If we presuppose, in a similar way, that $f(x_1 l)$ and $\overline{f}(x_1 l)$ are developed in the Mac Laurin series, we obtain the relations:

$$\overline{R}_N(x_1 l) = R_N(x_1 l), \quad (30a$$

$$\overline{f}(0) = f(0) = \varphi(0), \quad (30b$$

$$\frac{1}{U_\infty}\sum_{k=0}^{t-1} \frac{l^k}{t-k}(w_{t-k-1,k} - \overline{w}_{t-k-1,k}) = \frac{l^t}{t!}[\overline{f}^{(t)}(0) - f^{(t)}(0)],\ t = 1,2,\ldots,N \quad (30c$$

The relations (30a) show that the development in series of the arbitrary functions $f(x_1 l)$ and $\overline{f}(x_1 l)$ have to have the same remainder as order N.

From (30b) results that these functions must have the free term the same and equal with $\varphi(0)$, as well as the results from (30b) and (11a), and the relations (30c) can be interpreted as relations of connection between arbitrary functions $f(x_2)$ and $\overline{f}(x_2)$ and therefore can be eliminated among the conditions of connection of the studied problem of extremum.

b) The condition of canceling the axis of disturbance velocity u along the subsonic leading edge OA_1' (y=L, u=0) leads to the following relations of connection:

$$\sum_{q=0}^{E\left(\frac{n}{2}\right)} A'_{n,2q} L^{2q} = 0, \qquad n = 1,2, \ldots, N. \qquad (31)$$

If we mark with Θ'_{tj} and $\bar{\Theta}'_{tj}$ the expressions

$$\Theta'_{tj} = \sum_{q=0}^{E\left(\frac{t}{2}\right)} L^{2q} a'^{(t)}_{2q,j}, \qquad (32a)$$

$$\bar{\Theta}'_{tj} = \sum_{q=0}^{E\left(\frac{t}{2}\right)} L^{2q} \bar{a}'^{(t)}_{2q,j}, \qquad (32b)$$

then the relations (24) can be written under the form

$$F_1^{(t)} = \sum_{j=0}^{t-1} (\Theta'_{tj} w_{t-j-1,j} + \bar{\Theta}'_{tj} \bar{w}_{t-j-1,j}) = 0, \quad t = 1,2, \ldots, N. \qquad (33)$$

4.2. THE EXPLAINATION OF THE EXPRESSIONS OF THE AERODYNAMIC CHARACTERISTICS FOR THE UNFOLDED WING

As in the case of the folded wing, we have for explaination the following aerodynamic characterisitcs:

a) the lift power C_L;

b) the moment of dive C_M;

c) the advance resistance C_D;

d) the suction σ'.

a) The condition of the lift power of the wing C_L to be given ($C_L = C_{L0}$ = const) becomes, in the case of the unfolded wing, of the form

$$C_L = \frac{8}{U_\infty L} \int_{0CA'_1} u x_1 \, dx_1 \, dy =$$
$$= \sum_{n=1}^{N} \sum_{j=0}^{n-1} (\Lambda'_{nj} w_{n-j-1,j} + \bar{\Lambda}'_{nj} \bar{w}_{n-j-1,j}) = C_{L_0} = \text{const}. \qquad (34)$$

In this formula, the constants Λ'_{nj} have the values given in the appendix, and the constants $\bar{\Lambda}'_{nj}$ are obtained from Λ'_{nj} replacing the coefficients $a'^{(n)}_{2q\cdot j}$, $c'^{(n)}_{2q}$ etc. with their correspondenced barred letters.

b) in a similar way, the condition of the moment of dive to be given, $C_M = C_{M_0}$ = const, becomes, in the case of the unfolded wing, of the form

$$C_M = \frac{8}{U_\infty L}\int_{OCA'_1} u x_1^2 \,\mathrm{d}x_1\,\mathrm{d}y = $$
$$= \sum_{n=1}^{N}\sum_{j=0}^{n-1}\left(\Gamma'_{nj} w_{n-j-1,j} + \bar{\Gamma}'_{nj} \bar{w}_{n-j-1,j}\right) = C_{M_0} = \text{const.} \qquad (35)$$

In this expression, the constants Γ'_{nj} and $\bar{\Gamma}'_{nj}$ are connected by the constants Λ'_{nj} and $\bar{\Lambda}'_{nj}$ from formula (27) through the relations:

$$\Gamma'_{nj} = \frac{n+1}{n+2}\Lambda'_{nj}, \quad \bar{\Gamma}'_{nj} = \frac{n+1}{n+2}\bar{\Lambda}'_{nj}. \qquad (36)$$

c) The expression of the coefficient of wave resistance becomes, in the case of the unfolded wing, of the form

$$C_D = \frac{8}{U_\infty^2 L}\left[\int_{OA_1C} w u x_1 \,\mathrm{d}x_1\,\mathrm{d}y + \int_{OA_1A'_1} \bar{w} u x_1 \,\mathrm{d}x_1\,\mathrm{d}y\right] =$$
$$= \sum_{n=1}^{N}\sum_{m=1}^{N}\sum_{k=0}^{m-1}\sum_{j=0}^{n-1}\left[\Omega_{nmkj} w_{m-k-1,k} w_{n-j-1,j} + \Omega^{(1)}_{nmkj} \bar{w}_{m-k-1,k} w_{n-j-1,j} +\right.$$
$$\left. + \bar{\Omega}_{nmkj} w_{m-k-1,k} \bar{w}_{n-j-1,j} + \bar{\Omega}^{(1)}_{nmkj} \bar{w}_{m-k-1,k} \bar{w}_{n-j-1,j}\right]. \qquad (37)$$

The constants Ω_{nmkj} and $\Omega^{(1)}_{nmkj}$ are given in the appendix, and the constants $\bar{\Omega}_{nmkj}$ and $\bar{\Omega}^{(1)}_{nmkj}$ are deduced from Ω_{nmkj}, from $\Omega^{(1)}_{nmkj}$ respectively, replacing the coefficients $a'^{(n)}_{2q\cdot j}$, $c'^{(n)}_{2q\cdot j}$ etc. with their barred correspondences.

d) The coefficient of suction σ' is, similar with (9), (1),

$$\sigma' = -\frac{4\pi}{U_\infty^2 L}\sum_{n=1}^{N}\sum_{m=1}^{N}\frac{1}{m+n}\lim_{y\to L}\frac{\sqrt{1-B_1^2L^2}}{L}(L-y)\,u_{m-1}\,u_{n-1} \tag{38}$$

and can be written under the form (1)

$$\sigma' = \sum_{n=1}^{N}\sum_{m=1}^{N}\sum_{j=0}^{n-1}\sum_{k=0}^{m-1}\left[\Delta_{nmkj}\,w_{m-k-1.k}\,w_{n-j-1.j} + \Delta^{(1)}_{nmkj}\,\overline{w}_{m-k-1.k}\,w_{n-j-1.j} + \right.$$

$$\left. + \overline{\Delta}_{nmkj}\,w_{m-k-1.k}\,\overline{w}_{n-j-1.j} + \overline{\Delta}^{(1)}_{nmkj}\,\overline{w}_{m-k-1.k}\,\overline{w}_{n-j-1.j}\right], \tag{39}$$

where

$$\Delta_{nmkj} = -\frac{4\pi\sqrt{1-B^2L^2}}{U_\infty^2(m+n)}\sum_{q=0}^{E\left(\frac{n}{2}\right)}\sum_{q'=0}^{E\left(\frac{m}{2}\right)} L^{2(q+q')}\,a'^{(n)}_{2q.j}\,a'^{(m)}_{2q'.k}, \tag{40}$$

and the constants $\Delta^{(1)}_{nmkj}$, $\overline{\Delta}_{nmkj}$ and $\overline{\Delta}^{(1)}_{nmkj}$ are obtained replacing in Δ_{nmkj}, in turn, one of the coefficients $a'^{(n)}_{2q.j}$, $a'^{(m)}_{2q'.k}$ or respectively, both coefficients with their barred correspondences.

4.3 SOLUTION OF THE TWO VARIATIONAL PROBLEMS FOR THE UNFOLDED WING

We have now explained all the terms which enter into the problem of extremum of the unfolded wing. Coefficients $a'^{(n)}_{2q.j}$, $\bar{a}'^{(n)}_{2q.j}$ etc. are functions of the design of the folded and unfolded wings, that is 1 and L, as well as Mach number M_1 of the second cruising speed. Thus, these terms are constants in the studied problem of extremum.

Similarly, coefficients $\bar{w}_{n-j-1.j}$ of w, which have been determined from the first variational problem refering to the folded wing, are constants in the second problem of extremum.

Therefore, it results that we have to determine the coefficients $\bar{w}_{n-j-1.j}$ of $\bar{w}$ in such a way that, for the unfolded wing, the expression

$$C_D + \sigma' = \text{minim}, \tag{41}$$

at the cruising speed defined by Mach number M_1, with the conditions of connection

$$C_L = \text{const}, \qquad C_M = \text{const}, \; F_1^{(t)} = 0, \qquad t = 1, 2, \ldots, N. \tag{42}$$

This problem of extremum with connections is reduced to the calculation of the extremum without connections of expression

$$H = (C_D + \sigma') + \lambda^{(1)} C_L + \lambda^{(2)} C_M + \sum_{t=1}^{N} \lambda_t F_1^{(t)}. \tag{43}$$

We have explained all the terms which enter into the hamiltonian expression. Calculating its first variation and canceling it, we obtain the extrmum of H. If we then keep in mind that the coefficients $\bar{w}_{\theta\sigma}$ are independent, we will obtain, through the cancelation of the variations $\delta\bar{\omega}_{\theta\sigma}$, equations of the form

$$\sum_{n=1}^{N} \sum_{j=0}^{n-1} [(\bar{\Omega}^{(1)}_{n,\theta+\sigma+1.\sigma.j} + \bar{\Omega}^{(1)}_{\theta+\sigma+1.n.j.\sigma} + \bar{\Delta}^{(1)}_{n.\theta+\sigma+1.\sigma.j} + \bar{\Delta}^{(1)}_{\theta+\sigma+1.n.j.\sigma})\, \bar{w}_{n-j-1.j} +$$
$$+ (\Omega^{(1)}_{n.\theta+\sigma+1.\sigma.j} + \bar{\Omega}_{\theta+\sigma+1.n.j.\sigma} + \Delta^{(1)}_{n.\theta+\sigma+1.\sigma.j} + \bar{\Delta}_{\theta+\sigma+1.n.j.\sigma})\, w_{n-j-1.j}] +$$
$$+ \lambda^{(1)} \bar{\Lambda}_{\theta+\sigma+1.\sigma} + \lambda^{(2)} \bar{\Gamma}_{\theta+\sigma+1.\sigma} + \lambda_{\theta+\sigma+1} \bar{\Theta}_{\theta+\sigma+1.\sigma} = 0, \tag{44}$$
$$(\theta, \sigma) = 0, 1, \ldots, (N-1),$$
$$1 \leqslant \theta + \sigma + 1 \leqslant N.$$

These equations, together with the relations of connection (35), make up a liniar system of algebraic equations, which determine in a certain way the optimum values of the coefficients $\bar{w}_{n-j-1.j}$ of $\bar{w}$, as well as the values of the multiplicators $\lambda^{(1)}$, $\lambda^{(2)}$, and λ_t.

The problem of determining the form of the surface of the unfolded delta wing, optimum at the cruising speed, characterized by Mach number M_1, can be thus considered solved.

5.4 Calculation Appendix

We will mark with I_k the following indefinite integral:

$$I_k = \int \frac{y^k \, dy}{\sqrt{B_1 (L^2 - y^2)}} = -\frac{y^{k-1}}{k B_1^2} \sqrt{B_1^2 (L^2 - y^2)} + \frac{k-1}{k} L^2 I_{k-2} =$$

$$= \sum_{j=0}^{k} \frac{(-1)^j (k!) (2j)! \, L^{k-j}}{(j!)^3 (k-j)! \, B_1} \left[\left(\frac{L}{2} \right)^j \arcsin \frac{y}{L} + \right.$$

$$\left. + \frac{\sqrt{B_1^2 (L^2 - y^2)}}{B_1} \sum_{\nu=1}^{j} \frac{[(j-\nu)!]^2 L^{\nu-1}}{2^\nu (2j - 2\nu + 1)!} (L-y)^{j-\nu} \right] \tag{45}$$

and respectively with $\mathcal{J}'_k(\alpha, \beta)$ and $\mathcal{J}'^{**}_k(\alpha, \beta)$ the following definite integrals of type I_k:

$$\mathcal{J}'_k(\alpha, \beta) = \int_\alpha^\beta \frac{y^k \, dy}{\sqrt{B_1^2 (L^2 - y^2)}} \tag{46}$$

$$\mathcal{J}'^{**}_k(\alpha, \beta) = \int_\alpha^\beta \frac{y^k \, dy}{\sqrt{1 - B_1^2 y^2}} . \tag{47}$$

These definite integrals can be obtained taking the indefinite integral given in (45) between the limits α and β and making, for the the integral $\mathcal{J}_k^{**}(\alpha, \beta)$ the substitution $L = \frac{1}{B1}$.

With these notations, the expressions Λ_{nj} from formula (34) of the coefficient of lift power C_L will be of the form

$$\Lambda'_{nj} = \frac{8}{U_\infty L (n+1)} \left\{ \sum_{q=0}^{E\left(\frac{n}{2}\right)} a'^{(n)}_{2q,j} \mathcal{J}'_{2q}(0, L) + B_1 L \sum_{q=1}^{E\left(\frac{n-1}{2}\right)} \frac{c'^{(n)}_{2q,j}}{2q+1} \mathcal{J}'_{2q}(0, L) + \right.$$

$$\left. + \frac{\sqrt{B_1^2 (L^2 - l^2)}}{2} \sum_{q=0}^{n-1} \sum_{t=0}^{q} \frac{[1 + (-1)^t] \, l^{q-t}}{q+1} \mathcal{J}'_t(0, L) \right\} \tag{48}$$

if the unfolded wing has both supersonic leading edges and, respectively,

$$\Lambda_{nj}'^{**} = \frac{8}{U_\infty L(n+1)} \left\{ \sum_{q=0}^{E\left(\frac{n}{2}\right)} a_{2q.j}'^{(n)} \mathcal{J}_{2q}'^{**}\left(0, \frac{1}{B_1}\right) + \sum_{q=1}^{E\left(\frac{n-1}{2}\right)} \frac{c_{2q.j}'^{(n)}}{2q+1} \mathcal{J}_{2q}'^{**}\left(0, \frac{1}{B_1}\right) + \right.$$

$$\left. + \frac{\sqrt{1 - B^2 l_1^2}}{2} \sum_{q=0}^{n-1} \sum_{t=1}^{q} \frac{g_{qj}'^{(n)}}{q+1} [1 + (-1)^t] l^{q-t} \mathcal{J}_t'^{**}\left(0, \frac{1}{B_1}\right) \right\} \tag{49}$$

if the unfolded wing has both supersonic leading edges and the edges (provided from the edges of the folded wing) are subsonic.

If the edges OA_1 and OA_2 become supersonic, then in the expression (46) $\sqrt{1-B_1^2 l^2}$ is substituted with $\sqrt{B_1^2 l^2 - 1}$ in terms which contain coefficients $g_{qj}'^{(n)}$ and which represent only the contribution of these edges in the expression of $\Lambda_{nj}'^{**}$.

In a similar way, the expressions Ω_{nmkj} and $\Omega_{nmkj}^{(1)}$ from formula (37) of the coefficient of wave resistance of the unfolded wing will be, respectively, of the form

$$\Omega_{nmkj} = \frac{8}{U_\infty^2 L(m+n)} \left\{ \sum_{q=0}^{E\left(\frac{n}{2}\right)} a_{2q.j}'^{(n)} \mathcal{J}_{k+2q}'(0, l) + \right.$$

$$+ \sum_{q=1}^{E\left(\frac{n-1}{2}\right)} \frac{c_{2q.j}'^{(n)}}{k+2q+1} \left[B_1 L \mathcal{J}_{k+2q}'(0, l) + l^{k+2q} \operatorname{argch} \sqrt{\frac{L^2}{l^2}} \right] +$$

$$\left. + \frac{\sqrt{B_1^2 (L^2 - l^2)}}{2} \sum_{q=0}^{n-1} \sum_{t=0}^{k+q} \frac{g_{qj}'^{(n)}}{k+q+1} [1 + (-1)^{k+t}] l^{k+q-t} \mathcal{J}_t'(0, l) \right\} \tag{50}$$

and

$$\Omega^{(1)}_{nmkj} = \frac{8}{U^2_\infty L(m+n)} \left\{ \sum_{q=0}^{E\left(\frac{n}{2}\right)} a'^{(n)}_{2q,j} \mathcal{J}'_{k+2q}(l, L) + \right.$$

$$+ \sum_{q=1}^{E\left(\frac{n-1}{2}\right)} \frac{c'^{(n)}_{2q,j}}{k+2q+1} \left[B_1 L \mathcal{J}'_{k+2q}(l, L) - l^{k+2q} \operatorname{argch} \sqrt{\frac{L^2}{l^2}} \right] + \tag{51}$$

$$\left. + \frac{\sqrt{B_1^2(L^2 - l^2)}}{2} \sum_{q=0}^{n-1} \sum_{t=0}^{k+q} \frac{g'^{(n)}_{qj}}{k+q+1} [1 + (-1)^{k+t}] l^{k+q-t} \mathcal{J}'_t(l, L) \right\}$$

for the thin unfolded delta wing with both subsonic leading edges.

If the unfolded wing has supersonic leading edges and the subsonic edges, we will have the expressions Ω^{**}_{nmkj} and $\Omega^{**(1)}_{nmkj}$, respectively, of the form

$$\left. + 2(-1)^q [l^{k+q+1} - (-L)^{k+q+1}] \arccos \sqrt{\frac{(1 + B_1 L)(1 + B_1 l)}{2B_1(L + l)}} \right\rangle \Bigg\} \tag{52}$$

and

$$\Omega^{**(1)}_{nmkj} = \frac{8}{U^2_\infty L(m+n)} \left\{ \sum_{q=0}^{E\left(\frac{n}{2}\right)} a'^{(n)}_{2q,j} \mathcal{J}'^{**}_{k+2q}\left(l, \frac{1}{B_1}\right) + \right.$$

$$+ \sum_{q=0}^{E\left(\frac{n-1}{2}\right)} \frac{c'^{(n)}_{2q,j}}{k+2q+1} \left\langle \mathcal{J}'^{**}_{k+2q}\left(l, \frac{1}{B_1}\right) - l^{k+2q} \operatorname{argch} \sqrt{\frac{1}{B_1^2 l^2}} \right\rangle +$$

$$+ \sum_{q=0}^{n-1} \frac{g'^{(n)}_{qj}}{2(k+q+1)} \left\langle \sqrt{1 - B_1^2 l^2} \sum_{t=0}^{k+q} [1 + (-1)^{k+t}] l^{k+q-t} \mathcal{J}'^{**}_t\left(l, \frac{1}{B_1}\right) + \right.$$

$$\left. + 2(-1)^{q+1} [1 + (-1)^{k+q}] l^{k+q+1} \operatorname{argch} \sqrt{\frac{(1 + B_1 l)^2}{4B_1 l}} \right\rangle +$$

$$\Omega^{**}_{nmkj} = \frac{8}{U^2_\infty L(m+n)} \left\{ \sum_{q=0}^{E\left(\frac{n}{2}\right)} a'^{(n)}_{2q,j} \mathcal{J}'^{**}_{k+2q}(0, l) + \right.$$

$$+ \sum_{q=1}^{E\left(\frac{n-1}{2}\right)} \frac{c'^{(n)}_{2q,j}}{k+2q+1} \left\langle \mathcal{J}'^{**}_{k+2q}(0, l) + l^{k+2q} \operatorname{argch} \sqrt{\frac{1}{B^2 l^2}} \right\rangle +$$

$$+ \sum_{q=0}^{n-1} \frac{g_{qj}^{\prime(n)}}{2(k+q+1)} \left\{ \sqrt{1 - B_1^2 l_1^2} \sum_{t=0}^{k+q} [1 + (-1)^{k+t}] l^{k+q-t} \mathcal{J}_t^{\prime **}(0, l) + \right.$$

$$+ [1 + (-1)^{k+1}] l^{k+q+1} \operatorname{argch} \sqrt{\frac{1}{B_1^2 l^2}} +$$

$$\left. + 2(-1)^q [1 + (-1)^{k+q}] l^{k+q+1} \operatorname{argch} \sqrt{\frac{(1 + B_1 l)^2}{4 B_1 l}} \right\} +$$

$$+ \sum_{q=0}^{n-1} \frac{h_{qj}^{\prime(n)}}{2(k+q+1)} \left\{ \sqrt{B_1^2 L^2 - 1} \sum_{t=0}^{k+q} [1 + (-1)^{k+t}] L^{k+q-t} \mathcal{J}_t^{\prime **}(0, l) + \right.$$

$$+ [1 + (-1)^{k+1}] L^{k+q+1} \arccos \sqrt{\frac{1}{B_1^2 L^2}} -$$

$$- 2 (L^{k+q+1} - l^{k+q+1}) \arccos \sqrt{\frac{(1 + B_1 L)(1 - B_1 l)}{2 B_1 (L - l)}} +$$

$$+ \sum_{q=0}^{n-1} \frac{h_{qj}^{\prime(n)}}{2(k+q+1)} \left\{ \sqrt{B_1^2 L^2 - 1} \sum_{t=0}^{k+q} [1 + (-1)^{k+t}] L^{k+q-t} \mathcal{J}_t^{\prime **}\left(l, \frac{1}{B_1}\right) + \right.$$

$$+ 2 (L^{k+q+1} - l^{k+q+1}) \arccos \sqrt{\frac{(1 + B_1 L)(1 - B_1 l)}{2 B_1 (L - l)}} +$$

$$\left. + 2(-1)^{q+1} [l^{k+q+1} - (-L)^{k+q+1}] \arccos \sqrt{\frac{(1 + B_1 L)(1 + B_1 l)}{2 B_1 (L + l)}} \right\} \right\}. \quad (53)$$

If the edges and the leading edges of the unfolded wing are supersonic, we will have, respectively, the expressions:

$$\Omega_{nmkj}^{**} = \frac{8}{U_\infty^2 L (m+n)} \left\{ \sum_{q=0}^{E\left(\frac{n}{2}\right)} a_{2q,j}^{\prime(n)} \mathcal{J}_{k+2q}^{\prime **}\left(0, \frac{1}{B_1}\right) + \right.$$

$$+ \sum_{q=1}^{E\left(\frac{n-1}{2}\right)} \frac{c_{2q,j}^{\prime(n)}}{k + 2q + 1} \mathcal{J}_{k+2q}^{\prime **}\left(0, \frac{1}{B_1}\right) +$$

$$+ \sum_{q=0}^{n-1} \frac{g_{qj}^{\prime(n)}}{2(k+q+1)} \left\{ \sqrt{B_1^2 l^2 - 1} \sum_{t=0}^{k+q} [1 + (-1)^{k+t}] l^{k+q-t} \mathcal{J}_t^{\prime **}\left(0, \frac{1}{B_1}\right) + \right.$$

$$\left. + [1 + (-1)^{k+1}] l^{k+q+1} \arccos \sqrt{\frac{1}{B_1^2 l^2}} \right\} +$$

$$+\sum_{q=0}^{n-1}\frac{h_{qj}^{\prime(n)}}{2(k+q+1)}\left\{\sqrt{B_1^2L^2-1}\sum_{t=0}^{k+q}[1+(-1)^{k+t}]L^{k+q-t}\mathcal{J}_t^{\prime**}\left(0,\frac{1}{B_1}\right)+\right.$$

$$\left.+[1+(-1)^{k+1}]L^{k+q+1}\arccos\sqrt{\frac{1}{B_1^2L^2}}-\pi(L^{k+q+1}-l^{k+q+1})\right\} \quad (54)$$

and

$$\Omega_{nmkj}^{**(1)}=\frac{4\pi}{U_\infty^2L(m+n)}\sum_{q=0}^{n-1}\frac{L^{k+q+1}-l^{k+q+1}}{k+q+1}(g_{qj}^{\prime(n)}+h_{qj}^{\prime(n)}). \quad (55)$$

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